Relation Theory in Categories

By

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To Dr. George E. Strecker, without whose tactful proddings, infinite patience in proofreadings of handwritten drafts and helpful suggestions this work would never have been completed.

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The purpose of this dissertation has been to systematically generalize relation theory to a category theoretic context. A quite general relation theory has emerged which is applicable not only to concrete categories other than the category of sets and functions, but also to abstract categories whose objects need have no elements at all. This categorical approach has provided the opportunity to comprehend classical relation theory from a new vantage point, thus hopefully leading to an eventual better understanding of the subject.

A relation from an object X to an object Y is a pair (R,j) where j is an extremal monomorphism having domain R and codomain X×Y. By choosing j to be an extremal monomorphism, relations in the category of sets are the usual subsets of the Cartesian product, relations in the category of groups are subgroups of the group theoretic product, and relations in the category of topological spaces are subspaces of the topological product. This latter fact would not be the case if relations would be defined to be merely subobjects of the categorical product.

Section 0 notes results which are purely categorical in nature and which will be used extensively throughout the sequel. Particular emphasis is given to the epi-extremal mono factorization property and

necessary and sufficient conditions for the existence of this factorization and equivalent forms of the property.

In Section 1, the basic machinery for categorical relation theory is developed. For example, such notions as inverse relation, reflexive relation, symmetric relation, and composition of relations are defined and several important results are obtained.

Section 2 deals with a categorical definition of a congruence relation. Several algebraic results of Lambek and Cohn are generalized.

Equivalence relations and quasi-equivalence relations (symmetric, transitive relations) are studied in Section 3. A quasi-equivalence on an object X is shown to be an equivalence relation on a subobject of X.

If R is a set theoretic relation from the set X to the set Y and A is a subset of X then $AR = \{y\varepsilon Y$: there exists $a\varepsilon A$ such that $(a,y)\varepsilon R\}$. This definition is generalized in Section 4 and results similar to those obtained by Riguet are demonstrated.

If $\{(R_i,j_i): i\varepsilon I\}$ is a (finite) family of relations from X to Y then the relation theoretic union $(\bigcup_{i\varepsilon I} R_i, j)$ of the family is obtained by taking the intersection of all relations from X to Y which "contain" each R_i. If the category being investigated is assumed to have (finite) coproducts then the union of the family considered as subobjects and the relation theoretic union of the family considered as extremal subobjects turn out to be given by the unique extremal epi-mono and unique epi-extremal mono factorizations of the canonical morphism from the coproduct of the family to $X\times Y$.

The notion of a (finite) union distributive category is introduced. Roughly speaking, this property guarantees that unions "commute" with products and intersections.

Section 5 deals with unions and the importance of the concept of difunctional relation is brought out.

A well known result in set theoretic relation theory is that a partition determines an equivalence relation. In order to obtain this result in its generalized form the existence of an initial object which behaves similarly to the initial object in the category of sets (namely the empty set) is postulated and disjointness becomes a useful categorical notion. Also the notion of difunctional relations was crucial in obtaining the above result.

Section 6 deals with rectangular relations and the above result about partitions is obtained.

INTRODUCTION

The purpose of this work has been an attempt to systematically generalize relation theory to a category theoretic context. In doing so, several goals have been realized. Firstly, a quite general relation theory has emerged which is applicable not only to concrete categories other than the category of sets and functions, but also to abstract categories whose objects need have no elements at all. Secondly, taking a categorical approach has provided the opportunity to comprehend classical relation theory from a new vantage point, thus hopefully leading to an eventual better understanding of the subject.

Many relation theoretic results have been rather straightforward to prove in an "element free" setting, once the appropriate machinery has been constructed to handle them. On the other hand it has been surprising to see that some results which are easy to prove in the set theoretic context are much more difficult to show categorically.

For example, it is easy to prove that if R is a set theoretic relation from X to Y such that $RY = X$ then $RoR^{-1} = \{(x,z)$: there exists $y \varepsilon Y$ such that $(x,y) \varepsilon R$ and $(y,z) \varepsilon R^{-1}\}$ is reflexive. This result can be generalized to categories but is no longer easy to prove and the result gains some significance.

Another easy result in set theoretic relation theory is that if Δ_X and Δ_Y are the diagonals on X and Y respectively then $\Delta_X oR = R = Ro\Delta_Y$. This result is also generalized to categories but "isomorphic as rela-

tions" replaces "equality" and the result is no longer easy to prove.

Whenever one is generalizing properties care must be taken to be certain that the generalized definitions are really generalizations of the notions being considered and that the proper generalization of the definition is obtained. This seems to be particularly important in category theory. Care has been taken when selecting the basic notion of a relation from an object X to an object Y to be an extremal subobject of the categorical product X×Y; i.e. a pair (R,j) where j is an extremal monomorphism having domain R and codomain X×Y. By doing so relations in the category of sets are the usual subsets of the cartesian product, relations in the category of groups are subgroups of the group theoretic product, and relations in the category of topological spaces are subspaces of the topological product. This latter fact would not be the case if relations would be defined to be merely subobjects of the categorical product. Much care has also been taken with the definition of composition of relations (1.26). Using this definition many nice results have been obtained; however, in general, the composition of relations is not associative (1.36). This, at first glance, seems to be pathological and casts doubt on the suitability of the definition of composition of relations. However, the wealth of other important results obtained belies this doubt (see 1.37). Also, some further atonement is yielded by the fact that for rectangular relations composition <u>is</u> associative (6.15).

Cohn [3] and Lambek [13] define a congruence in an algebraic setting to be a subalgebra of the cartesian product which is "compatible" with the algebraic operations and which is set theoretically an equivalence relation. In this work, a generalized notion of congruence

is given which is equivalent to the above in algebraic categories and the result that a (categorical) congruence is a (categorical) equivalence relation is obtained.

It was found that categorical unions were very difficult to work with. However, by assuming the category being studied had (finite) coproducts as well as being locally small and quasi-complete the notion of union became somewhat easier to handle.

For instance, if $\{(R_i, j_i) : i \in I\}$ is a (finite) family of relations from X to Y then the union $(\bigcup_{i \in I} R_i, j)$ of the family, considered as subobjects of X×Y is not necessarily a relation from X to Y, since j is not necessarily an extremal monomorphism. The relation theoretic union of the family is obtained by taking the unique epi-extremal mono factorization of j (5. 3) or equivalently by taking the intersection of all relations from X to Y which "contain" each R_i. If the category being investigated is assumed to have (finite) coproducts in addition to being locally small and quasi-complete then the union of the family considered as subobjects and the relation theoretic union of the family considered as extremal subobjects turn out to be given by the unique extremal epi-mono and unique epi-extremal mono factorizations of the canonical morphism from the coproduct of the family to X×Y (5.29). It is also shown that when the category has (finite) coproducts both factorizations respect unions (5.30 and 5.42).

Unions are still difficult to handle even with the assumption of (finite) coproducts mentioned above; hence, the notion of a (finite) union distributive category is introduced (5.31). Roughly speaking, this property guarantees that unions "commute" with products and intersections and thus unions become "easy" to handle. Examples of union

distributive categories show that such categories tend to be more of a topological nature rather than of an algebraic nature.

The set theoretic notion of difunctional relation is due to Riguet [22] and its importance has been noted by Lambek [13] and MacLane [18]. A set theoretic relation R is difunctional if and only if $RoR^{-1}oR \subset R$. The categorical definition in view of the fact that associativity cannot be assumed reads: R is difunctional if and only if $(RoR^{-1})oR \leq R$ and $Ro(R^{-1}oR) \leq R$ where "$\leq$" is the usual order on sub-objects. It is easy to prove, again by choosing elements, that if a set theoretic relation R is difunctional then $R = RoR^{-1}oR$. However, the similar result in the categorical setting is much harder to obtain and is rephrased: if R is difunctional then $R \equiv (RoR^{-1})oR$ and $R \equiv Ro(R^{-1}oR)$ where "$\equiv$" means isomorphic as extremal subobjects (5.28).

A well known result in set theoretic relation theory is that a partition determines an equivalence relation. In order to obtain this result in its generalized form additional hypotheses had to be added to the category being studied. In particular, the existence of an initial object which behaves similarly to the initial object in the category of sets (namely the empty set) had to be postulated and disjointness became a useful categorical notion. Again, examples of such categories are non-algebraic. Also the notion of difunctional relations was crucial in obtaining the above result (6.20).

The excellent reference paper by Riguet [22] has been used as a guide for the results of set theoretic relation theory. Indeed, most all of the results contained herein are generalizations of results in [22]. The papers by Lambek [13], [14], MacLane [18] and Bednarek and Wallace [1], [2] provided motivation for many of the generali-

zations.

The basis for the categorical notions has been taken from the papers of Herrlich and Strecker [7], [8], Isbell [11], [12], and the forthcoming text by Herrlich and Strecker [9] (which has greatly influenced this work). For most of the basic categorical notions the reader is referred to the texts by Mitchell [21], Freyd [4] and Herrlich and Strecker [9].

The work here is begun with a preliminary Section 0 which notes (often without proof) results which are purely categorical in nature and which will be used extensively throughout the sequel. Particular emphasis is given to the epi-extremal mono factorization property and necessary and sufficient conditions for the existence of this factorization and equivalent forms of the property. However, it is not intended that the preliminary section give a complete category-theoretical background. It is expected that the reader be familiar with the basic categorical notions.

SECTION 0. PRELIMINARIES

0.0. Remark. It is assumed that the reader is familiar with the basic notions of category theory and hence such basic notions as epimorphism, monomorphism, retraction, section, equalizer, regular monomorphism, coequalizer, regular epimorphism, subobject and limits shall not be defined. The reader is referred to Mitchell [21] and Herrlich and Strecker [9] for such notions. All of the following results are proved in detail in Herrlich and Strecker [9] . Since Theorem 0.21 is vital to this work the proof is sketched here.

0.1. Notation. The category whose class of objects is the class of all sets and whose morphism class is the class of all functions shall be denoted by Set.

The category whose class of objects is the class of all groups and whose morphism class is the class of all group homomorphisms shall be denoted by Grp.

The category whose class of objects is the class of all topological spaces and whose morphism class is the class of all continuous functions shall be denoted by Top_1.

In a manner similar to that described above, one obtains the following categories:

FSet - finite sets and functions;

FGp - finite groups and group homomorphisms;

Ab - Abelian groups and group homomorphisms;

$\underline{SGp}$ - semigroups and semigroup homomorphisms;

$\underline{SGp}^1$ - semigroups with identity and semigroup homomorphisms which preserve the identity;

$\underline{Rng}$ - rings and ring homomorphisms;

$\underline{Rng}^1$ - rings with identity and ring homomorphisms which preserve the identity;

$\underline{Top}_2$ - Hausdorff spaces and continuous functions;

$\underline{CpT}_2$ - compact Hausdorff spaces and continuous functions.

0.2. Proposition. Let $\mathcal{C}$ be a category and let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be $\mathcal{C}$-morphisms.

1) If f and g are monomorphisms then gf is a monomorphism.

2) If f and g are epimorphisms then gf is an epimorphism.

3) If gf is a monomorphism then f is a monomorphism.

4) If gf is an epimorphism then g is an epimorphism.

5) If gf is an isomorphism then g is a retraction and f is a section.

0.3. Remark. In general, an equalizer is a limit of a certain diagram. It is an object together with a morphism whose domain is the object. A regular monomorphism is a morphism for which there exists a diagram so that the domain of the morphism together with the morphism is the equalizer of the diagram.

It is observed in Herrlich and Strecker [9] that certain functors preserve regular monomorphisms while not preserving equalizers, hence one reason for the above distinction between equalizers and regular monomorphisms.

In this paper, since we shall not deal with functors, no distinc-

tion shall be made between equalizers and regular monomorphisms; i.e., between the pair (object and morphism) and the morphism alone. Both will be called equalizers.

0.4. Proposition. Let $\mathcal{C}$ be a category and let $X \xrightarrow{f} Y$ be a $\mathcal{C}$-morphism. Then the following are equivalent:

1) f is an isomorphism,

2) f is a monomorphism and a retraction,

3) f is an epimorphism and a section,

4) f is a monomorphism and a regular epimorphism,

5) f is an epimorphism and a regular monomorphism.

0.5. Definition. Let $\{A_i\colon i\varepsilon I\}$ be a family of $\mathcal{C}$-objects then the product $(\prod_{i\varepsilon I} A_i, \pi_i)$ of the family is a $\mathcal{C}$-object $\prod_{i\varepsilon I} A_i$ together with projection morphisms $\pi_i\colon \prod_{i\varepsilon I} A_i \longrightarrow A_i$ with the property that if P is any $\mathcal{C}$-object for which there exist $\mathcal{C}$-morphisms $\rho_i\colon P \longrightarrow A_i$ for each $i\varepsilon I$, then there exists a unique morphism $\lambda\colon P \longrightarrow \prod_{i\varepsilon I} A_i$ such that $\pi_i \lambda = \rho_i$ for each $i\varepsilon I$.

The dual notion is that of the coproduct $(\coprod_{i\varepsilon I} A_i, \mu_i)$.

0.6. Definition. Let $\{(A_i,a_i)\colon i\varepsilon I\}$ be a family of subobjects of a $\mathcal{C}$-object X. Then the intersection $(\bigcap_{i\varepsilon I} A_i, a)$ of the family is a $\mathcal{C}$-object $\bigcap_{i\varepsilon I} A_i$ together with a morphism $a\colon \bigcap_{i\varepsilon I} A_i \longrightarrow X$ where for each i there is a morphism $\lambda_i\colon \bigcap_{i\varepsilon I} A_i \longrightarrow A_i$ such that $a_i\lambda_i = a$ with the property that if P is any object for which there exist $\mathcal{C}$-morphisms $\rho\colon P \longrightarrow X$ and $\rho_i\colon P \longrightarrow A_i$ such that $a_i\rho_i = \rho$ for each $i\varepsilon I$ then there exists a unique morphism $\lambda\colon P \longrightarrow \bigcap_{i\varepsilon I} A_i$ such that $a\lambda = \rho$.

It follows that a is a monomorphism.

0.7. Remark. The above two definitions are mentioned because of the fundamental role they play in the sequel. They are special limits and are perhaps the most important limits in the categories that will be considered in this work.

The following theorem is a special case of a more general theorem dealing with the commutation of limits which can be found in Herrlich and Strecker [9]. A variation of the theorem will be proved in Section 1 (1.5).

0.8. Theorem. Let $\{(A_i,a_i): i\varepsilon I\}$ and $\{(B_i,b_i): i\varepsilon I\}$ be families of sub-objects of $\mathcal{C}$ -objects X and Y respectively. Then if $\mathcal{C}$ has finite products and arbitrary intersections then $(\bigcap_{i\varepsilon I} A_i)\times(\bigcap_{i\varepsilon I} B_i)$ and $\bigcap_{i\varepsilon I}(A_i\times B_i)$ are canonically isomorphic.

0.9. Notation. Let $\{X_i: i\varepsilon I\}$ be a family of $\mathcal{C}$ -objects and suppose $\{Z \xrightarrow{f_i} X_i: i\varepsilon I\}$ is a family of $\mathcal{C}$ -morphisms. Then by the definition of product there exists a unique morphism h from Z to $\prod_{i\varepsilon I} X_i$ such that $\pi_i h = f_i$ for each $i\varepsilon I$. This morphism h shall be denoted by $\langle f_i\rangle_{i\varepsilon I}$.

Let A and B be $\mathcal{C}$ -objects and suppose that $a: A \longrightarrow X$ and $b: B \longrightarrow Y$ are $\mathcal{C}$ -morphisms. If ρ_1 and ρ_2 are the projection morphisms from $A\times B$ to A and B respectively then $a\rho_1: A\times B \longrightarrow X$ and $b\rho_2: A\times B \longrightarrow Y$, hence by the definition of product there exists a unique morphism g from $A\times B$ to $X\times Y$ such that $\pi_1 g = a\rho_1$ and $\pi_2 g = b\rho_2$. This morphism g shall be denoted by $a\times b$ and shall be called the product of a and b.

Let f be a $\mathcal{C}$ -morphism from X to Y. If f is a monomorphism then the following notation shall be used:

$$X \rightarrowtail^{f} Y$$

If f is an epimorphism then the following notation shall be used:

$$X \overset{f}{\twoheadrightarrow} Y$$

If f is an equalizer then the following notation shall be used:

$$X \rightarrowtail\!\!\!\!\!\!\!\rightarrowtail\!\!\xrightarrow{f} Y$$

If f is an isomorphism then the following notation shall be used:

$$X \rightarrowtail\!\!\!\xrightarrow{f}\!\!\!\twoheadrightarrow Y$$

0.10. Proposition. Let $A \xrightarrow{a} X$, $B \xrightarrow{b} Y$, $X \xrightarrow{c} Z$, and $Y \xrightarrow{d} W$ be $\mathcal{C}$-morphisms. Then $(c\times d)\ (a\times b) = ca\times db$.

0.11. Proposition. Let $A \xrightarrow{a} X$ and $B \xrightarrow{b} Y$ be monomorphisms (respectively, sections, isomorphisms) then $a\times b$ is a monomorphism (section, isomorphism).

0.12. Remark. A partial order may be defined on the subobjects of an object in $\mathcal{C}$ in the following way:

If X is a $\mathcal{C}$-object and (A,a) and (B,b) are subobjects of X; i.e., a and b are monomorphisms with codomain X and domains A and B respectively, then $(A,a) \leq (B,b)$ if and only if there exists a morphism c from A to B such that $bc = a$.

$$\begin{tikzcd}
B \arrow[rr, tail, "b"] & & X \\
A \arrow[u, "c"] \arrow[urr, tail, "a"'] & &
\end{tikzcd}$$

By an abuse of language, if $(A,a) \leq (B,b)$ then (B,b) is said to contain (A,a) and the morphism c is sometimes called the inclusion of (A,a) into (B,b). It is easy to see that if $(A,a) \leq (B,b)$ and $(B,b) \leq (A,a)$ then the morphism c is an isomorphism. In this case,

(A,a) and (B,b) are said to be <u>isomorphic as subobjects of</u> X. This is a stronger condition than A and B just being isomorphic objects in the category $\mathcal{C}$. The following notation shall be used to denote the case where (A,a) and (B,b) are isomorphic as subobjects of X:

$$(A,a) \equiv (B,b).$$

Sometimes it is written (inaccurately) that $A \leq B$ or that A and B are isomorphic as subobjects of X. When this is done, the morphisms a and b should be clear from the context.

It is immediate that $(A,a) \equiv (B,b)$ if and only if $(A,a) \leq (B,b)$ and $(B,b) \leq (A,a)$. Thus the relation "$\leq$" on subobjects is easily seen to be a partial order up to isomorphism as subobjects.

0.13. <u>Definition</u>. Let f from X to Y be a $\mathcal{C}$-morphism. f is an <u>extremal monomorphism</u> if and only if f is a monomorphism and whenever f = gh and h is an epimorphism then h is an isomorphism.

If f is an extremal monomorphism the following notation shall be used:

$$X \overset{f}{\rightarrowtail\!\!\!\to} Y$$

The dual notion is that of an <u>extremal epimorphism</u> and is denoted:

$$X \overset{f}{\twoheadrightarrow\!\!\!\to} Y$$

If f is an extremal monomorphism $f\colon X \rightarrowtail\!\!\!\to Y$, then (X,f) is called an <u>extremal subobject</u> of Y.

0.14. <u>Remark</u>. The definition of extremal monomorphism is due to Isbell [11]. The concept of extremal monomorphism is important since it yields what shall be called the "image" of a morphism (see 0.18).

0.15. <u>Examples</u>. In the categories <u>Set</u>, <u>Grp</u>, <u>Ab</u> and <u>FGp</u>, extremal monomorphisms are precisely the monomorphisms (i.e., one-to-one morphisms).

In the categories $\underline{Top_1}$ and $\underline{CpT_2}$ extremal monomorphisms are precisely the embeddings. In the category $\underline{Top_2}$ they are the closed embeddings.

0.16. Proposition. If $X \xrightarrow{f} Y$ is a $\mathcal{C}$-morphism such that $f = gh$ and f is an extremal monomorphism then h is an extremal monomorphism.

0.17. Proposition. If $X \xrightarrow{f} Y$ is a $\mathcal{C}$-morphism then the following are equivalent:

1) f is an isomorphism,

2) f is an epimorphism and an extremal monomorphism,

3) f is a monomorphism and an extremal epimorphism (c.f. 0.3).

0.18. Definition. A category $\mathcal{C}$ is said to have the unique epi-extremal mono factorization property if for any $\mathcal{C}$-morphism $X \xrightarrow{f} Y$, there exist an epimorphism h and an extremal monomorphism g with $f = gh$ such that whenever $f = g'h'$ where g' is an extremal monomorphism and h' is an epimorphism then there exists an isomorphism σ such that the following diagram commutes.

$$
\begin{tikzcd}
X \arrow[rrrr, "f"] \arrow[drr, two heads, "h"] \arrow[ddrr, two heads, "h'"'] & & & & Y \\
& & Z \arrow[urr, tail, "g"'] \arrow[d, "\sigma"] & & \\
& & Z' \arrow[uurr, tail, "g'"'] & &
\end{tikzcd}
$$

If $\mathcal{C}$ has the unique epi-extremal mono factorization property and if $f = gh$ where h is an epimorphism and g is an extremal monomorphism, then the pair (h,g) shall be used to designate the epi-extremal mono factorization of f. The extremal subobject (Z,g) of Y is called the

image of X under f. Sometimes (Z,g) is referred to as the *image of f*.

The notion of the *unique extremal epi-mono factorization property* is defined dually.

If $\mathfrak{C}$ has the unique extremal epi-mono factorization property and $f = gh$ where g is a monomorphism and h is an extremal epimorphism then the pair (h,g) shall be used to designate the extremal epi-mono factorization of f. The subobject (Z,g) of Y is called the *subimage of X under f*. Sometimes (Z,g) is referred to as the *subimage* of f.

0.19. *Definition*. A category $\mathfrak{C}$ is said to have the *diagonalizing property* if whenever $gh = ab$ such that h is an epimorphism and a is an extremal monomorphism, then there exists a (necessarily unique) morphism ξ such that $\xi h = b$ and $a\xi = g$.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "b"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\xi"'] \\ W \arrow[r, "a"'] & Z \end{tikzcd}$$

0.20. *Theorem*. Let $\mathfrak{C}$ be a locally small category having equalizers and intersections. Then the following are equivalent:

1) $\mathfrak{C}$ has the unique epi-extremal mono factorization property,

2) $\mathfrak{C}$ has the diagonalizing property,

3) the intersection of extremal monomorphisms is an extremal monomorphism and the composite of extremal monomorphisms is an extremal monomorphism,

4) if $\mathfrak{C}$ has pullbacks and if (P,α,β) is the pullback of f and g where $f\beta = g\alpha$ and f is an extremal monomorphism then α is an extremal

monomorphism.

5) if $\mathcal{C}$ has (finite) products then the (finite) product of extremal monomorphisms is an extremal monomorphism.

0.21. <u>Theorem</u>. If $\mathcal{C}$ is locally small and has equalizers and intersections then $\mathcal{C}$ has both the unique epi-extremal mono factorization property and the unique extremal epi-mono factorization property.

<u>Proof</u>. (sketch). First we will show the existence of the unique extremal epi-mono factorization property. If f from X to Y is any $\mathcal{C}$-morphism then let $(\bigcap_{j\varepsilon J} E_j, e)$ be the intersection of the family $\{(E_j, e_j): j\varepsilon J\}$ of all subobjects of Y through which f factors. Then it follows that e is a monomorphism and that f factors through e; i.e., there exists a morphism h such that f = eh. Now, to see that h is an epimorphism suppose α and β are $\mathcal{C}$-morphisms such that $\alpha h = \beta h$. Let (E,k) be the equalizer of α and β. It follows from the definition of equalizer that there exists a morphism g such that kg = h.

$$\begin{tikzcd}
 & X \arrow[rr, "f"] \arrow[dl, "g"'] \arrow[dr, "h"] & & Y \\
E \arrow[rr, "k"'] & & \bigcap_{j\varepsilon J} E_j \arrow[ur, "e"'] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Z
\end{tikzcd}$$

Thus it follows that f factors through ek and since ek is a monomorphism then there exists a morphism $\lambda: \bigcap_{j\varepsilon J} E_j \longrightarrow E$ such that $ek\lambda = e$. From this it follows that k is an isomorphism whence $\alpha = \beta$ and so h is an epimorphism.

Next it will be shown that h is an extremal epimorphism. Suppose $h = h_1 h_2$ where h_1 is a monomorphism. Then eh_1 is a monomorphism through

which f factors. From this it follows, as above, that h_1 is an isomorphism and hence h is an extremal epimorphism. Suppose $f = g'h'$ where g' is a monomorphism and h' is an extremal epimorphism. Then since g' is a monomorphism through which f factors there exists a morphism τ from $\bigcap_{j\varepsilon J} E_j$ to the codomain of h' (domain of g') such that $e = g'\tau$. Since e and g' are monomorphisms, it follows that $h' = \tau h$ and that τ is a monomorphism. Since h' is an extremal epimorphism it follows that τ is an isomorphism. Thus $\mathcal{E}$ has the unique extremal epi-mono factorization property.

Now suppose that $\bar{g}\bar{e} = m\bar{h}$ where $\bar{e}$ is an epimorphism and m is an extremal monomorphism. It will be shown that there exists a morphism $\bar{\sigma}$ from the codomain of $\bar{e}$ to the domain of m such that $\bar{\sigma}\bar{e} = h$ and $m\bar{\sigma} = g$.

Let $(\bigcap_{i\varepsilon I} A_i, a)$ be the intersection of the family $\{(A_i, a_i): i\varepsilon I\}$ of all subobjects of the codomain of $\bar{g}$ (codomain of m) through which $\bar{g}$ and m factor. This family is non-empty since both $\bar{g}$ and m factor through the identity morphism on the codomain of $\bar{g}$. It follows that both $\bar{g}$ and m factor through a. Thus there exist morphisms a_1 and a_2 such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[rr, two heads, "\bar{e}"] \arrow[dd, "\bar{h}"'] & & Y \arrow[dd, "\bar{g}"] \arrow[dl, "a_1"'] \\
& \bigcap_{i\varepsilon I} A_i \arrow[dr, "a"] & \\
W \arrow[ur, "a_2"] \arrow[rr, tail, "m"'] & & Z
\end{tikzcd}$$

It will be shown next that a_2 is an epimorphism. Suppose α^* and β^* are $\mathcal{E}$-morphisms for which $\alpha^* a_2 = \beta^* a_2$. Let (b^*, k^*) be the equalizer

of α^* and β^*. It follows from the definition of equalizer that there exists a morphism b_1 such that $k^*b_1 = a_2$, since $\alpha^*a_2 = \beta^*a_2$. Since the diagram commutes it follows that $\alpha^*a_1\bar{e} = \beta^*a_1\bar{e}$. But $\bar{e}$ is an epimorphism hence $\alpha^*a_1 = \beta^*a_1$ so that by the definition of equalizer there exists a morphism b_2 such that $k^*b_2 = a_1$. Thus it follows that $m = ak^*b_1$ and $\bar{g} = ak^*b_2$ and so both m and $\bar{g}$ factor through ak^* from which it follows that k^* is an isomorphism. Hence $\alpha^* = \beta^*$ and a_2 is an epimorphism. But m is an extremal monomorphism and $m = aa_2$ and a_2 is an epimorphism. Thus a_2 is an isomorphism. Thus defining $\bar{\sigma} = a_2^{-1}a_1$ it follows that the following diagram commutes and $\mathcal{C}$ has the diagonalization property.

$$\begin{tikzcd}
X \arrow[r, two heads, "\bar{e}"] \arrow[d, "\bar{h}"'] & Y \arrow[d, "\bar{g}"] \arrow[dl, "\bar{\sigma}"'] \\
W \arrow[r, tail, "m"'] & Z
\end{tikzcd}$$

Hence $\mathcal{C}$ has the unique epi-extremal mono factorization property (0.20).

0.22. <u>Theorem</u>. Let $\mathcal{C}$ be any category then the following are equivalent:

1) $\mathcal{C}$ is (finitely) complete,

2) $\mathcal{C}$ has (finite) products and (finite) intersections,

3) $\mathcal{C}$ has (finite) products and equalizers,

4) $\mathcal{C}$ has (finite) products and pullbacks.

0.23. <u>Definition</u>. A category $\mathcal{C}$ is said to be <u>quasi-complete</u> if $\mathcal{C}$ has finite products and arbitrary intersections.

0.24. Examples. The categories FSet and FGp are quasi-complete categories which are not complete. The categories Set, $\underline{Top}_1$, $\underline{Top}_2$, $\underline{CpT}_2$, Grp, Ab, Ring, and SGp are quasi-complete.

0.25. Remarks. A quasi-complete category is finitely complete but is not necessarily complete as the examples FSet and FGp above show.

Also, a locally small, quasi-complete category has both the unique extremal epi-mono factorization property and the unique epi-extremal mono factorization property (0.20 and 0.21).

It can be shown that the unique epi-extremal mono factorization of a morphism can be obtained by taking the intersection of all extremal monomorphisms through which the morphism factors. It has been shown that the unique extremal epi-mono factorization property is obtained by taking the intersection of all subobjects through which the morphism factors (0.20). These characterizations shall be used frequently in the sequel.

SECTION 1. GENERALITIES

1.0. *Standing Hypothesis. Throughout the entire paper it will be assumed that $\mathcal{C}$ is a locally small, quasi-complete (finite products and arbitrary intersections) category.*

As noted in the preliminary section $\mathcal{C}$ enjoys the unique epi – extremal mono factorization property.

1.1. *Examples*. Many well known categories are locally small, and quasi-complete. Among such are the categories: *Set*, $\underline{\mathrm{Top}_1}$, $\underline{\mathrm{Top}_2}$, *Grp*, *Ab*, *SGp*, $\underline{\mathrm{SGp}^1}$, *Rng*, $\underline{\mathrm{Rng}^1}$, $\underline{\mathrm{CpT}_2}$, and *FGp*.

1.2. *Definition*. Let X and Y be $\mathcal{C}$ -objects. A *relation* R *from* X *to* Y is an extremal subobject of X×Y; i.e., a relation from X to Y is a pair (R,j) where R is a $\mathcal{C}$ -object and j is an extremal monomorphism having domain R and codomain X×Y. A relation from X to X is called a *relation on* X.

1.3. *Definition*. Let (R,j) and (S,k) be relations from X to Y. Then (R,j) and (S,k) are said to be *isomorphic relations* if and only if they are isomorphic as extremal subobjects of X×Y.

1.4. *Examples*. In the categories *Set*, and $\underline{\mathrm{Top}_1}$ relations are subsets of the Cartesian product together with the inclusion map.

In the categories *Grp*, and *Ab* relations are subgroups of the Cartesian product together with the inclusion map.

In the categories $\underline{Top_2}$, and $\underline{CpT_2}$, relations are closed subspaces of the Cartesian product together with the inclusion map.

1.5. Proposition. Let X and Y be $\mathcal{B}$-objects and let (A,a) and (B,b) be extremal subobjects of Y. Then $X\times(A\cap B)$ and $(X\times A)\cap(X\times B)$ are isomorphic relations from X to Y.

Proof. Consider the following commutative diagrams.

$$\begin{tikzcd}
A\cap B \arrow[r, "\lambda_B"] \arrow[d, "\lambda_A"'] \arrow[dr, "c"] & B \arrow[d, "b"] \\
A \arrow[r, "a"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\times A)\cap(X\times B) \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] \arrow[dr, "\gamma_2"] & X\times B \arrow[d, "1_X\times b"] \\
X\times A \arrow[r, "1_X\times a"'] & X\times Y
\end{tikzcd}$$

Consider also $(X\times(A\cap B),\ 1_X\times c = \gamma_1)$. Since extremal subobjects are closed under intersections and products (0.20) γ_1 and γ_2 are extremal monomorphisms.

Since $(1_X\times a)(1_X\times\lambda_A) = 1_X\times c = \gamma_1$ and $(1_X\times b)(1_X\times\lambda_B) = 1_X\times c = \gamma_1$ then by the definition of intersection there exists a unique morphism σ from $X\times(A\cap B)$ to $(X\times A)\cap(X\times B)$ so that $\gamma_2\sigma = \gamma_1$ and the following diagram commutes. Thus:

$$(X\times(A\cap B),\ \gamma_1) \leq ((X\times A)\cap(X\times B),\ \gamma_2).$$

$$\begin{tikzcd}
 & & X\times A \arrow[dr, "1_X\times a"] & \\
X\times(A\cap B) \arrow[urr, "1_X\times\lambda_A"] \arrow[r, dashed, "\sigma"] \arrow[drr, "1_X\times\lambda_B"'] & (X\times A)\cap(X\times B) \arrow[ur, "\lambda_1"] \arrow[rr, "\gamma_2"] \arrow[dr, "\lambda_2"] & & X\times Y \\
 & & X\times B \arrow[ur, "1_X\times b"'] &
\end{tikzcd}$$

Now let (π_1,π_2), (π_1',π_2'), (ρ_1,ρ_2) and (ρ_1',ρ_2') be the projections of $X\times Y$, $X\times(A\cap B)$, $X\times A$, and $X\times B$ respectively. Observe that:

$$\pi_1\gamma_2 = \pi_1(1_X\times a)\lambda_1 = 1_X\rho_1\lambda_1 = \pi_1(1_X\times b)\lambda_2 = 1_X\rho_1'\lambda_2$$

$$\pi_2\gamma_2 = \pi_2(1_X\times a)\lambda_1 = a\rho_2\lambda_1 = \pi_2(1_X\times b)\lambda_2 = b\rho_2'\lambda_2.$$

Thus by the definition of intersection there exists a unique morphism Σ from $(X\times A)\cap(X\times B)$ to $A\cap B$ such that $c\Sigma = \pi_2\gamma_2$ and thus by the definition of product there exists a unique morphism ξ from $(X\times A)\cap(X\times B)$ to $X\times(A\cap B)$ such that $\xi = <\pi_1\gamma_2,\Sigma>$; i.e., $\pi_1'\xi = \pi_1\gamma_2$ and $\pi_2'\xi = \Sigma$. Now $\gamma_1\xi = (1_X\times c)\xi$ hence $\pi_1\gamma_1\xi = 1_X\pi_1'\xi = \pi_1\gamma_2$ and $\pi_2\gamma_1\xi = c\pi_2'\xi = c\Sigma = \pi_2\gamma_2$. Thus $\gamma_1\xi = \gamma_2$, whence:

$$((X\times A)\cap(X\times B),\ \gamma_2) \leq (X\times(A\cap B),\ \gamma_1).$$

$$\begin{tikzcd}
X & & & (X\times A)\cap(X\times B) \arrow[lll, "\pi_1\gamma_2"'] \arrow[dll, "\Sigma"'] \arrow[dlll, bend right, "\xi"'] \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] \arrow[dd, "\gamma_2"] & \\
 & A\cap B & X\times A \arrow[dr, "1_X\times a"'] & & X\times B \arrow[dl, "1_X\times b"] \\
X\times(A\cap B) \arrow[uu, "\pi_1'"] \arrow[ur, "\pi_2'"] \arrow[rrr, dashed, "\gamma_1"'] & & & X\times Y &
\end{tikzcd}$$

1.6. Notation. Let X and Y be $\mathcal{C}$ -objects and let $(X\times Y,\pi_1,\pi_2)$ and $(Y\times X,\rho_1,\rho_2)$ be the indicated products of X and Y. Then there exists a unique isomorphism from X×Y to Y×X, denoted by $<\pi_2,\pi_1>$, such that the following diagram commutes.

$$\begin{tikzcd}
X\times Y \arrow[rr, tail, two heads, "<\pi_2,\pi_1>"] \arrow[dd, "\pi_1"'] \arrow[d, "\pi_2"] & & Y\times X \arrow[d, "\rho_1"'] \arrow[dd, "\rho_2"] \\
Y \arrow[rr, tail, two heads, "1_Y"] & & Y \\
X \arrow[rr, tail, two heads, "1_X"] & & X
\end{tikzcd}$$

Note that: $<\pi_2,\pi_1><\rho_2,\rho_1> = 1_{Y\times X}$ and $<\rho_2,\rho_1><\pi_2,\pi_1> = 1_{X\times Y}$.

1.7. Definition. Let (R,j) be a relation from X to Y and let (τ,j^*) be the unique epi-extremal mono factorization of $<\pi_2,\pi_1>j$ (see 0.18). The codomain of τ (domain of j^*) is denoted by R^{-1} and (R^{-1},j^*) is called the inverse relation of (R,j) or more simply, when there is little likelihood of confusion, the inverse of R.

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[dr, tail, two heads, "\tau"'] & X\times Y \arrow[r, tail, two heads, "<\pi_1,\pi_2>"] & Y\times X \\
& R \arrow[ur, tail, "j^*"'] &
\end{tikzcd}$$

1.8. Example. In the categories Set, $\underline{Top_1}$, $\underline{Top_2}$, Grp, Ab, and FGp,

$$<\pi_2,\pi_1>: X\times Y \longrightarrow Y\times X$$

is defined by $<\pi_2,\pi_1>(x,y) = (y,x)$; hence, if (R,j) is a relation from X to Y then $R^{-1} \cong \{(y,x): (x,y)\varepsilon R\}$ with j^* the inclusion map of R^{-1} into Y×X.

1.9. Proposition. If (R,j) is a relation from X to Y the R and R^{-1} are isomorphic objects of $\mathcal{C}$.

Proof. Since $<\pi_2,\pi_1>$ is an isomorphism and j is an extremal monomorphism then $<\pi_2,\pi_1>j$ is an extremal monomorphism. But $<\pi_2,\pi_1>j = j^*\tau$. Thus since τ is an epimorphism then from the definition of extremal monomorphism it follows that τ is an isomorphism.

1.10. Definition. If (R,j) is a relation from X to X then R is said to be symmetric if and only if $(R^{-1},j^*) \leq (R,j)$.

1.11. Proposition. Let (R,j) be a relation from X to Y. Then the inverse relation $((R^{-1})^{-1},j\#)$ of (R^{-1},j^*) and (R,j) are isomorphic relations.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[d, "\tau"'] & X\times Y \arrow[d, "<\pi_2,\pi_1>"] \\
R^{-1} \arrow[r, tail, "j^*"] \arrow[d, "\tau\#"'] & Y\times X \arrow[d, "<\rho_2,\rho_1>"] \\
(R^{-1})^{-1} \arrow[r, tail, "j\#"] & X\times Y
\end{tikzcd}$$

Since the two inner squares commute the outer rectangle commutes. Both of τ and $\tau\#$ have been shown to be isomorphisms (1.9). And, as also has been observed: $<\rho_2,\rho_1><\pi_2,\pi_1> = 1_{X\times Y}$ (1.6). Consequently, $\tau\#\tau$ is an isomorphism and $j = j\#(\tau\#\tau)$. Thus $(R,j) \equiv ((R^{-1})^{-1},j\#)$.

1.12. Proposition. Let (R,j) and (S,k) be relations from X to Y. Then $(S,k) \leq (R,j)$ if and only if $(S^{-1},k^*) \leq (R^{-1},j^*)$.

Proof. Consider the following commutative diagram.

$$
\begin{tikzcd}
R \arrow[r, "j"] \arrow[dr, "\tau"] & X\times Y \arrow[r, "<\pi_2,\pi_1>"] & Y\times X \\
 & R^{-1} \arrow[ur, "j^*"] & \\
S \arrow[uu, dashed, "\alpha"] \arrow[uur, "k"] \arrow[r, "\tilde{\tau}"] & S^{-1} \arrow[u, dashed, "\beta"'] \arrow[uur, "k^*"'] &
\end{tikzcd}
$$

If $(S,k) \leq (R,j)$ then there exists a morphism $\alpha: S \longrightarrow R$ such that $j\alpha = k$. Define $\beta = \tau\alpha\tilde{\tau}^{-1}$. Then $j^*\beta = j^*\tau\alpha\tilde{\tau}^{-1} = <\pi_2,\pi_1>j\alpha\tilde{\tau}^{-1} = <\pi_2,\pi_1>k\tilde{\tau}^{-1}$ $= k^*\tilde{\tau}\tilde{\tau}^{-1} = k^*$. Thus $(S^{-1},k^*) \leq (R^{-1},j^*)$.

If $(S^{-1},k^*) \leq (R^{-1},j^*)$ then by the above, $((S^{-1})^{-1},k\#) \leq ((R^{-1})^{-1},j\#)$ thus $(S,k) \leq (R,j)$ (1.11).

1.13. Corollary. If (R,j) is a symmetric relation on X then $(R,j) \leq (R^{-1},j^*)$ whence $(R,j) \equiv (R^{-1},j^*)$.

Proof. Since (R,j) is symmetric $(R^{-1},j^*) \leq (R,j)$. Thus

$$(R,j) \equiv ((R^{-1})^{-1},j\#) \leq (R^{-1},j^*) \text{ (1.11 and 1.12)}.$$

Consequently $(R,j) \equiv (R^{-1},j^*)$.

1.14. Definition. Recall that since $\mathcal{B}$ is quasi-complete it has equalizers, thus for each $\mathcal{B}$-object X let (Δ_X,i_X) denote the equalizer of π_1 and π_2 where π_1 and π_2 are the projections of $X\times X$. Since i_X is an equalizer it is an extremal monomorphism. Hence (Δ_X,i_X) is always a relation on X (called the diagonal of $X\times X$).

A relation (R,j) on X is said to be reflexive on X provided that $(\Delta_X,i_X) \leq (R,j)$.

1.15. Example. In the categories Grp, Ab, Set, Top_1, Top_2, and CpT_2, it follows that $\Delta_X \equiv \{(x,x): x\varepsilon X\} \subset X\times X$ with the inclusion map.

1.16. Proposition. For any $\mathcal{C}$ -object X, $<\pi_2,\pi_1>i_X = i_X$. Thus:

$$(\Delta_X, i_X) \equiv (\Delta_X^{-1}, i_X^*).$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X\times X \arrow[rr, "<\pi_2,\pi_1>"] & & X\times X \\
\Delta_X \arrow[u, "i_X"] \arrow[urr, "i_X"'] & &
\end{tikzcd}$$

$$\pi_1<\pi_2,\pi_1>i_X = \pi_2 i_X = \pi_1 i_X = \pi_2<\pi_2,\pi_1>i_X.$$

Thus the epi-extremal mono factorization of $<\pi_2,\pi_1>i_X$ is (i_{Δ_X}, i_X).

1.17. Corollary. Let (R,j) be a relation on X, then (R,j) is reflexive on X if and only if (R^{-1},j^*) is reflexive on X.

Proof. If $(\Delta_X,i_X) \leq (R,j)$ then $(\Delta_X,i_X) \equiv (\Delta_X^{-1},i_X^*) \leq (R^{-1},j^*)$ (1.16 and 1.12).

Conversely if $(\Delta_X,i_X) \leq (R^{-1},j^*)$ then $(\Delta_X,i_X) \equiv (\Delta_X^{-1},i_X^*) \leq ((R^{-1})^{-1},j\#) \equiv (R,j)$ (1.16, 1.12. and 1.11).

1.18. Proposition. Let (R,j) and (S,k) be relations from X to Y. Then the relations $(R\cap S)^{-1}$ and $(R^{-1}\cap S^{-1})$ are isomorphic relations.

Proof. According to the definitions of intersection and inverse relation we have the following commutative diagrams.

$$\begin{tikzcd}
 & R \arrow[dr, "j"] & \\
R\cap S \arrow[ur, "\lambda_1"] \arrow[rr, "\psi"] \arrow[dr, "\lambda_2"'] & & X\times Y \\
 & S \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R^{-1} \arrow[dr, "j^*"] & \\
R^{-1}\cap S^{-1} \arrow[ur, "\lambda_3"] \arrow[rr, "\phi"] \arrow[dr, "\lambda_4"'] & & Y\times X \\
 & S^{-1} \arrow[ur, "k^*"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
R\cap S \arrow[r, "\psi"] \arrow[dr, "\tau^*"'] & X\times Y \arrow[r, "\langle\pi_2,\pi_1\rangle"] & Y\times X \\
& (R\cap S)^{-1} \arrow[ur, "\psi^*"'] &
\end{tikzcd}$$

Observe that $\langle\pi_2,\pi_1\rangle\psi = j^*\tau\lambda_1$ and $\langle\pi_2,\pi_1\rangle\psi = k^*\tilde{\tau}\lambda_2$. Thus by the definition of intersection: $(R\cap S,\langle\pi_2,\pi_1\rangle\psi) \leq (R^{-1}\cap S^{-1},\phi)$. However since τ^* is an isomorphism, $(R\cap S,\langle\pi_2,\pi_1\rangle\psi) \equiv ((R\cap S)^{-1},\psi^*)$; whence

$$((R\cap S)^{-1},\psi^*) \leq (R^{-1}\cap S^{-1},\phi).$$

To obtain the reverse inequality, note that by the definition of intersection $(R^{-1}\cap S^{-1},\langle\rho_2,\rho_1\rangle) \leq (R\cap S,\psi)$ since $j\tau^{-1}\lambda_3 = \langle\rho_2,\rho_1\rangle\phi$ and $k\tilde{\tau}^{-1}\lambda_4 = \langle\rho_2,\rho_1\rangle\phi$. Thus $(R^{-1}\cap S^{-1},\langle\pi_2,\pi_1\rangle\langle\rho_2,\rho_1\rangle\phi) \leq (R\cap S,\langle\pi_2,\pi_1\rangle\psi)$. Whence $(R^{-1}\cap S^{-1},\phi) \leq (R\cap S,\langle\pi_2,\pi_1\rangle\psi) \equiv ((R\cap S)^{-1},\psi^*)$. Consequently:

$$(R^{-1}\cap S^{-1},\phi) \equiv ((R\cap S)^{-1},\psi^*).$$

1.19 <u>Remark</u>. It is clear from the definition of intersection (0.6) that if $(R,j) \leq (S,k)$ and $(R,j) \leq (T,m)$ then $(R,j) \leq (S\cap T,n)$.

1.20. <u>Proposition</u>. Let (R,j) be a relation on X. Then $R\cap\Delta_X$, $R^{-1}\cap\Delta_X$, and $R\cap R^{-1}\cap\Delta_X$ are isomorphic relations on X.

<u>Proof</u>. Consider the following commutative diagram.

$$\begin{tikzcd}
& & & & R^{-1} \arrow[dddr, "j^*"] & \\
& & & R^{-1}\cap\Delta_X \arrow[ur, "\lambda_4"'] \arrow[dr, "\lambda_3"] & & \\
& & & & \Delta_X \arrow[dl, "i_X"'] \arrow[dr, "i_X"] & \\
R\cap R^{-1}\cap\Delta_X \arrow[uuurrrr, bend left, "\lambda_7"] \arrow[r, "\lambda_5"] \arrow[ddrrrr, bend right, "\lambda_6"'] & R \arrow[uuurrr, bend left, "\tau"] \arrow[rr, "j"] & & X\times X \arrow[rr, "\langle\pi_2,\pi_1\rangle"] & & X\times X \\
& & R\cap\Delta_X \arrow[ul, "\lambda_1"] \arrow[drr, "\lambda_2"] & & & \\
& & & & \Delta_X \arrow[uul, "i_X"] \arrow[uur, "i_X"'] &
\end{tikzcd}$$

Note that since i_X equalizes π_1 and π_2, $<\pi_2,\pi_1>i_X = i_X$ (1.16) and also that $<\pi_2,\pi_1>^{-1} = <\pi_2,\pi_1>$; i.e., $<\pi_2,\pi_1><\pi_2,\pi_1> = 1_{X\times X}$ (1.6). Observe that $j\tau^{-1}\lambda_4 = <\pi_2,\pi_1>^{-1}j^*\lambda_4 = <\pi_2,\pi_1>^{-1}i_X\lambda_3 = <\pi_2,\pi_1>i_X\lambda_3$. Consequently $j\tau^{-1}\lambda_4 = <\pi_2,\pi_1>i_X\lambda_3 = i_X\lambda_3$. Thus by the definition of intersection:

$$(R^{-1}\cap\Delta_X,i_X\lambda_3) \leq (R\cap\Delta_X,i_X\lambda_2).$$

Also observe that $j^*\tau\lambda_1 = <\pi_2,\pi_1>j\lambda_1 = <\pi_2,\pi_1>i_X\lambda_2$. Whence $j^*\tau\lambda_1 = i_X\lambda_2$ so that by the definition of intersection:

$$(R\cap\Delta_X,i_X\lambda_2) \leq (R^{-1}\cap\Delta_X,i_X\lambda_3).$$

Thus:

$$(R\cap\Delta_X,i_X\lambda_2) \equiv (R^{-1}\cap\Delta_X,i_X\lambda_3).$$

Clearly $(R\cap R^{-1}\cap\Delta_X,i_X\lambda_6) \leq (R\cap\Delta_X,i_X\lambda_2)$. But $(R\cap\Delta_X,i_X\lambda_2) \leq (R^{-1}\cap\Delta_X,i_X\lambda_3)$ and $(R\cap\Delta_X,i_X\lambda_2) \leq (R\cap\Delta_X,i_X\lambda_2)$. Thus:

$$(R\cap\Delta_X,i_X\lambda_2) \leq (R\cap\Delta_X,i_X\lambda_2)\cap(R^{-1}\cap\Delta_X,i_X\lambda_3) \equiv (R\cap R^{-1}\cap\Delta_X,i_X\lambda_6).$$

Hence,

$$(R\cap\Delta_X,i_X\lambda_2) \equiv (R\cap R^{-1}\cap\Delta_X,i_X\lambda_6).$$

1.21. Lemma. If X is a $\mathcal{C}$-object and $X \xrightarrow{<1_X,1_X>} X\times X$ is the unique morphism h such that $\pi_1 h = \pi_2 h = 1_X$, then $(X,<1_X,1_X>)$ and (Δ_X,i_X) are isomorphic relations on X.

Proof. Since $\pi_1<1_X,1_X> = \pi_2<1_X,1_X>$ and i_X is the equalizer of π_1 and π_2, it follows that $(X,<1_X,1_X>) \leq (\Delta_X,i_X)$. Since $\pi_1<1_X,1_X> = 1_X$, $<1_X,1_X>$ is a section, hence an extremal monomorphism.

Clearly, $\pi_1<1_X,1_X>\pi_1 i_X = 1_X\pi_1 i_X = \pi_1 i_X$ and $\pi_2<1_X,1_X>\pi_1 i_X = 1_X\pi_1 i_X = \pi_1 i_X = \pi_2 i_X$. Hence, by the definition of product, $<1_X,1_X>\pi_1 i_X = i_X$. Thus $(\Delta_X,i_X) \leq (X,<1_X,1_X>)$.

1.22. Example. In the categories Set, Top_1, Top_2, Grp, Ab, and Rng, $<1_X,1_X>: X \longrightarrow X\times X$ can be defined by $<1_X,1_X>(x) = (x,x) \varepsilon X\times X$ for all $x\varepsilon X$.

1.23. Remark. It is also easy to see that up to isomorphism of extremal subobjects $(X,<1_X,1_X>)$ (and thus (Δ_X,i_X) also) is the equalizer of each of the following sets of morphisms:

$\{\pi_1,\pi_2\}$, $\{<1_X,1_X>\pi_1,\ <1_X,1_X>\pi_2\}$, $\{<1_X,1_X>\pi_1,\ 1_{X\times X}\}$, $\{<1_X,1_X>\pi_2,\ 1_{X\times X}\}$, and $\{<1_X,1_X>\pi_1, <1_X,1_X>\pi_2,\ 1_{X\times X}\}$.

1.24. Proposition. If (R,j) is a reflexive relation on X then $\pi_1 j$ and $\pi_2 j$ are retractions.

Proof. Since $(X,<1_X,1_X>) \leq (\Delta_X,i_X) \leq (R,j)$ there exist morphisms α and β such that $i_X\alpha = <1_X,1_X>$ and $j\beta = i_X$. Thus $1_X = \pi_1<1_X,1_X> = \pi_1 i_X\alpha = \pi_1 j\beta\alpha$. Thus $\pi_1 j$ is a retraction. Similarly $\pi_2 j$ is a retraction.

1.25. Remark. Consider the following products: $(X\times Y,\rho_1,\rho_2)$, $(Y\times Z,\tilde{\rho}_1,\tilde{\rho}_2)$, $(X\times Y\times Z,\bar{\pi}_1,\bar{\pi}_2,\bar{\pi}_3)$, $((X\times Y)\times Z,\tilde{\pi}_1,\tilde{\pi}_2)$ and $(X\times(Y\times Z),\pi_1{}^*,\pi_2{}^*)$. It is easy to see there exist isomorphisms

$$\Theta_1 = <\rho_1\tilde{\pi}_1,\rho_2\tilde{\pi}_1,\tilde{\pi}_2> \text{ and } \Theta_2 = <\pi_1{}^*,\tilde{\rho}_1\pi_2{}^*,\tilde{\rho}_2\pi_2{}^*>$$

$$\begin{tikzcd}(X\times Y)\times Z \arrow[r, "\Theta_1"] & X\times Y\times Z & X\times(Y\times Z) \arrow[l, "\Theta_2"']\end{tikzcd}$$

such that $\bar{\pi}_1\Theta_1 = \rho_1\tilde{\pi}_1$, $\bar{\pi}_2\Theta_1 = \rho_2\tilde{\pi}_1$, $\bar{\pi}_3\Theta_1 = \tilde{\pi}_2$ and $\bar{\pi}_1\Theta_2 = \pi_1{}^*$, $\bar{\pi}_2\Theta_2 = \tilde{\rho}_1\pi_2{}^*$, and $\bar{\pi}_3\Theta_2 = \tilde{\rho}_2\pi_2{}^*$.

1.26. Definition. Let (R,j) be a relation from X to Y and (S,k) be a relation from Y to Z. Consider the following intersection.

$$\begin{tikzcd}
& R\times Z \arrow[r, "1\times j"] & (X\times Y)\times Z \arrow[dr, "\Theta_1"] & \\
R\times Z\cap X\times S \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] \arrow[rrr, "\gamma"] & & & X\times Y\times Z \\
& X\times S \arrow[r, "1\times k"'] & X\times(Y\times Z) \arrow[ur, "\Theta_2"'] &
\end{tikzcd}$$

Let $<\bar{\pi}_1,\bar{\pi}_3>$ denote that unique morphism from $X\times Y\times Z$ to $X\times Z$ such that $\sigma_1<\bar{\pi}_1,\bar{\pi}_3> = \bar{\pi}_1$ and $\sigma_2<\bar{\pi}_1,\bar{\pi}_3> = \bar{\pi}_3$ where σ_1 and σ_2 are the projections of $X\times Z$ to X and Z respectively.

Let (τ',j') be the unique epi-extremal mono factorization of $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma$, and let the codomain of τ' (domain of j') be denoted by RoS. The relation (RoS,j') is called the <u>composition of</u> R <u>and</u> S.

1.27. <u>Examples</u>. In the categories <u>Set</u>, <u>Grp</u>, <u>Ab</u>, and $\underline{Top_1}$, the composition of R and S is isomorphic to the set

$\{(x,y)$: there exists a $y\varepsilon Y$ such that $(x,y)\varepsilon R$ and $(y,z)\ \varepsilon S\}$.

This is the usual set theoretic composition of relations (which incidentally is not the usual notation for the composition of functions when they are considered as relations).

In the category $\underline{Top_2}$, the composition of R and S is the closure of the above set.

1.28. <u>Definition</u>. If (R,j) is a relation on X then R is said to be <u>transitive</u> if and only if $(RoR,j') \leq (R,j)$.

A relation on an object X is said to be an <u>equivalence relation</u> if and only if it is reflexive, symmetric, and transitive.

1.29. <u>Examples</u>. In the categories <u>Set</u> and $\underline{Top_1}$, transitive relations and equivalence relations are the usual set theoretic transitive relations and equivalence relations together with the inclusion maps.

In the category $\underline{Top_2}$, equivalence relations are closed set theoretic equivalence relations.

In the categories <u>Grp</u>, and <u>Ab</u>, equivalence relations are subgroups of the catesian product which are set theoretic equivalence relations.

1.30. <u>Proposition</u>. Let (R_1,j_1) and (R_2,j_2) be relations from X to Y and let (S_1,k_1) and (S_2,k_2) be relations from Y to Z and suppose $(R_1,j_1) \leq (R_2,j_2)$ and $(S_1,k_1) \leq (S_2,k_2)$. Then $(R_1oS_1,j) \leq (R_2oS_2,k)$.

Proof. Since $(R_1,j_1) \leq (R_2,j_2)$ and $(S_1,k_1) \leq (S_2,k_2)$ it is immediate that $(R_1\times Z, j_1\times 1) \leq (R_2\times Z, j_2\times 1)$ and $(X\times S_1, 1\times k_1) \leq (X\times S_2, 1\times k_2)$ whence $((R_1\times Z)\cap(X\times S_1),\gamma_1) \leq ((R_2\times Z)\cap(X\times S_2),\gamma_2)$. Consequently there exists a morphism α such that the following diagram commutes.

$$
\begin{tikzcd}
 & R_2\times Z \arrow[r, tail, "j_2\times 1"] & (X\times Y)\times Z \arrow[ddr, "\Theta_1"] & \\
 & R_1\times Z \arrow[ur, tail, "j_1\times 1"] & & \\
(R_2\times Z)\cap(X\times S_2) \arrow[uur] \arrow[ddr] \arrow[rrr, "\gamma_2"] & & & X\times Y\times Z \\
(R_1\times Z)\cap(X\times S_1) \arrow[u, "\alpha"] \arrow[uur] \arrow[ddr] \arrow[urrr, tail, "\gamma_1"'] & & & \\
 & X\times S_2 \arrow[dr, tail, "1\times k_2"] & & \\
 & X\times S_1 \arrow[r, tail, "1\times k_1"'] & X\times(Y\times Z) \arrow[uuur, "\Theta_2"'] &
\end{tikzcd}
$$

Thus $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma_2\alpha = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma_1$.

Since $(R_1 o S_1, j)$ is the intersection of all extremal subobjects through which $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma_1$ factors (0.21) and since $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma_1$ factors through $(R_2 o S_2, k)$ it follows that $(R_1 o S_1, j) \leq (R_2 o S_2, k)$ which was to be proved.

1.31. Theorem. Let (R,j) be a relation from X to Y then $R o \Delta_Y$, R, and $\Delta_X o R$ are isomorphic relations from X to Y.

Proof. First consider $R o \Delta_Y$. From the definition of composition of relations the following commutative diagram is obtained.

$$
\begin{tikzcd}
 & R\times Y \arrow[r, tail, "j\times 1"] & (X\times Y)\times Y \arrow[dr, "\Theta_1"] & \\
(R\times Y)\cap(X\times\Delta_Y) \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] \arrow[rrr, tail, "\gamma"] & & & X\times Y\times Y \\
 & X\times\Delta_Y \arrow[r, tail, "1\times i_Y"'] & X\times(Y\times Y) \arrow[ur, "\Theta_2"'] &
\end{tikzcd}
$$

Recall that (Δ_Y, i_Y) is the equalizer of the projections ρ_1 and ρ_2 from $Y\times Y$ to Y.

It will next be shown that $\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \langle\bar\pi_1,\bar\pi_3\rangle\gamma$. Let $\tilde\rho_1$, and $\tilde\rho_2$ be the projections of $X\times\Delta_Y$ to X and Δ_Y respectively, and let $\pi_1{}^*$ and $\pi_2{}^*$ be the projections of $X\times(Y\times Y)$ to X and $Y\times Y$ respectively. Then

$\rho_1\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_1\gamma = \rho_1\langle\bar\pi_1,\bar\pi_3\rangle\gamma$.

$\rho_2\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_2\gamma = \bar\pi_2\Theta_2(1\times i_Y)\lambda_2 = \rho_1\pi_2{}^*(1\times i_Y)\lambda_2 = \rho_1 i_Y\tilde\rho_2\lambda_2 = \rho_2 i_Y\tilde\rho_2\lambda_2 =$
$\rho_2\pi_2{}^*(1\times i_Y)\lambda_2 = \bar\pi_3\Theta_2(1\times i_Y)\lambda_2 = \bar\pi_3\gamma = \rho_2\langle\bar\pi_1,\bar\pi_3\rangle\gamma$.

Hence $\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \langle\bar\pi_1,\bar\pi_3\rangle\gamma$.

Let $\bar\rho_1$ and $\bar\rho_2$ be the projections of $(X\times Y)\times Y$ to $X\times Y$ and Y respectively and let $\rho_1{}^*$ and $\rho_2{}^*$ be the projections of $R\times Y$ to R and Y respectively.

Since $\pi_1 j\rho_1{}^*\lambda_1 = \pi_1\bar\rho_1(j\times 1)\lambda_1 = \bar\pi_1\Theta_1(j\times 1)\lambda_1 = \bar\pi_1\gamma = \pi_1\langle\bar\pi_1,\bar\pi_2\rangle\gamma$, and $\pi_2 j\rho_1{}^*\lambda_1 = \pi_2\bar\rho_1(j\times 1)\lambda_1 = \bar\pi_2\Theta_1(j\times 1)\lambda_1 = \bar\pi_2\gamma = \pi_2\langle\bar\pi_1,\bar\pi_2\rangle\gamma$, and $\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \langle\bar\pi_1,\bar\pi_3\rangle\gamma = j'\tau'$, then the following diagram commutes.

$$\begin{tikzcd}
 & R\times Y \arrow[rr, "\rho_1{}^*"] & & R \arrow[dr, "j"] & \\
(R\times Y)\cap(X\times\Delta_Y) \arrow[ur, "\lambda_1"] \arrow[rrrr, "\langle\bar\pi_1,\bar\pi_2\rangle\gamma"] \arrow[drr, "\tau'"'] & & & & X\times Y \\
 & & Ro\Delta_Y \arrow[urr, "j'"'] & &
\end{tikzcd}$$

Thus since $(Ro\Delta_Y, j')$ is the intersection of all extremal subobjects through which $\langle\bar\pi_1,\bar\pi_3\rangle\gamma = \langle\bar\pi_1,\bar\pi_2\rangle\gamma$ factors (0.21) it follows that $(Ro\Delta_Y, j') \le (R,j)$.

To see that $(R,j) \le (Ro\Delta_Y, j')$ consider the following commutative diagrams.

$$
\begin{tikzcd}
R \arrow[r, "{\langle j, \pi_2 j\rangle}"] \arrow[dr, "j"] \arrow[ddr, "\pi_2 j"'] & (X\times Y)\times Y \arrow[r, two heads, "\Theta_1"] \arrow[d, "\bar\rho_1"'] \arrow[dd, bend left, "\bar\rho_2"] & X\times Y\times Y \arrow[d, "{\langle\bar\pi_1, \bar\pi_2\rangle}"'] \arrow[dd, bend left, "\bar\pi_3"] \\
 & X\times Y \arrow[r, two heads, "1_{X\times Y}"] & X\times Y \\
 & Y \arrow[r, two heads, "1_Y"] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
R \arrow[r, "{\langle 1_R, \pi_2 j\rangle}"] \arrow[dr, "1_R"] \arrow[ddr, "\pi_2 j"'] & R\times Y \arrow[d, "\rho_1*"'] \arrow[dd, bend left, "\rho_2*"] \\
 & R \\
 & Y
\end{tikzcd}
$$

Recall that $(\Delta_Y, i_Y) \equiv (Y, \langle 1_Y, 1_Y\rangle)$ (1.21), thus there exists a morphism $\sigma : Y \longrightarrow \Delta_Y$ such that $i_Y\sigma = \langle 1_Y, 1_Y\rangle$.

$$
\begin{tikzcd}
R \arrow[r, "{\langle \pi_1 j, \sigma\pi_2 j\rangle}"] \arrow[dr, "\pi_1 j"] \arrow[ddr, "\sigma\pi_2 j"'] & X\times\Delta_Y \arrow[d, "\tilde\rho_1"'] \arrow[dd, bend left, "\tilde\rho_2"] \\
 & X \\
 & \Delta_Y
\end{tikzcd}
$$

It now will be shown that the following diagram commutes.

$$\begin{tikzcd}
 & R\times Y \arrow[r, tail, "j\times 1_Y"] & (X\times Y)\times Y \arrow[dr, "\theta_1"] & \\
R \arrow[ur, "{\langle 1_R,\pi_2 j\rangle}"] \arrow[rrr, tail, "{\theta_1\langle j,\pi_2 j\rangle}"] \arrow[dr, "{\langle \pi_1 j,\sigma\pi_2 j\rangle}"'] & & & X\times Y\times Y \\
 & X\times\Delta_Y \arrow[r, tail, "1_X\times i_Y"'] & X\times(Y\times Y) \arrow[ur, "\theta_2"'] &
\end{tikzcd}$$

$$\bar{\pi}_1\theta_1(j\times 1)\langle 1_R,\pi_2 j\rangle = \pi_1\bar{\rho}_1(j\times 1)\langle 1_R,\pi_2 j\rangle = \pi_1 j\rho_1{}^*\langle 1_R,\pi_2 j\rangle = \pi_1 j 1_R = \pi_1 j.$$

$$\bar{\pi}_2\theta_1(j\times 1)\langle 1_R,\pi_2 j\rangle = \pi_2\bar{\rho}_1(j\times 1)\langle 1_R,\pi_2 j\rangle = \pi_2 j\rho_1{}^*\langle 1_R,\pi_2 j\rangle = \pi_2 j 1_R = \pi_2 j.$$

$$\bar{\pi}_3\theta_1(j\times 1)\langle 1_R,\pi_2 j\rangle = \bar{\rho}_2(j\times 1)\langle 1_R,\pi_2 j\rangle = \rho_2{}^*\langle 1_R,\pi_2 j\rangle = \pi_2 j.$$

$$\bar{\pi}_1\theta_1\langle j,\pi_2 j\rangle = \pi_1\bar{\rho}_1\langle j,\pi_2 j\rangle = \pi_1 j.$$

$$\bar{\pi}_2\theta_1\langle j,\pi_2 j\rangle = \pi_2\bar{\rho}_1\langle j,\pi_2 j\rangle = \pi_2 j.$$

$$\bar{\pi}_3\theta_1\langle j,\pi_2 j\rangle = \bar{\rho}_2\langle j,\pi_2 j\rangle = \pi_2 j.$$

$$\bar{\pi}_1\theta_2(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle = \pi_1{}^*(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle = \tilde{\rho}_1\langle \pi_1 j,\sigma\pi_2 j\rangle = \pi_1 j.$$

$$\begin{aligned}\bar{\pi}_2\theta_2(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle &= \rho_1\pi_2{}^*(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle = \rho_1 i_Y\tilde{\rho}_2\langle \pi_1 j,\sigma\pi_2 j\rangle = \\ &\qquad \rho_1 i_Y\sigma\pi_2 j = \rho_1\langle 1_Y,1_Y\rangle\pi_2 j = 1_Y\pi_2 j = \pi_2 j.\end{aligned}$$

$$\begin{aligned}\bar{\pi}_3\theta_2(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle &= \rho_2\pi_2{}^*(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle = \rho_2 i_Y\tilde{\rho}_2\langle \pi_1 j,\sigma\pi_2 j\rangle = \\ &\qquad \rho_2 i_Y\sigma\pi_2 j = \rho_2\langle 1_Y,1_Y\rangle\pi_2 j = 1_Y\pi_2 j = \pi_2 j.\end{aligned}$$

Thus by the definition of intersection there exists a unique morphism ξ from R to $(R\times Y)\cap(X\times\Delta_Y)$ such that $\gamma\xi = \theta_1\langle j,\pi_2 j\rangle = \theta_1(j\times 1_Y)\langle 1_R,\pi_2 j\rangle = \theta_2(1\times i_Y)\langle \pi_1 j,\sigma\pi_2 j\rangle$. Thus $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\xi = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\theta_1\langle j,\pi_2 j\rangle$. But since $\pi_1\langle\bar{\pi}_1,\bar{\pi}_3\rangle\theta_1\langle j,\pi_2 j\rangle = \bar{\pi}_1\theta_1\langle j,\pi_2 j\rangle = \pi_1\bar{\rho}_1\langle j,\pi_2 j\rangle = \pi_1 j$ and $\pi_2\langle\bar{\pi}_1,\bar{\pi}_3\rangle\theta_1\langle j,\pi_2 j\rangle = \bar{\pi}_3\theta_1\langle j,\pi_2 j\rangle = \bar{\rho}_2\langle j,\pi_2 j\rangle = \pi_2 j$ it follows from the definition of product that $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\theta_1\langle j,\pi_2 j\rangle = j$.

Thus $j = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\xi = j'\tau'\xi$ whence $(R,j) \leq (R\circ\Delta_Y,j')$.

The proof that $(R,j) \equiv (\Delta_X\circ R,j'')$ follows from analogous arguments.

1.32. Proposition. If (R,j) is reflexive and transitive on X then

$$(R\circ R,j') \equiv (R,j).$$

Proof. Since (R,j) is transitive then $(R\circ R,j') \leq (R,j)$. Since (R,j) is reflexive then $(\Delta_X,i_X) \leq (R,j)$. Thus $(R,j) \leq (R\circ\Delta_X,j'') \leq (R\circ R,j') \leq (R,j)$ (1.31 and 1.30). Hence $(R\circ R,j') \equiv (R,j)$.

1.33. Remark. As has been remarked in (1.27), if (R,j) is a relation from X to Y and (S,k) is a relation from Y to Z then in the categories Set, Ab, Grp, and Top_1, $(R\circ S,j')$ may be taken to be the set

$$\{(x,z)\colon \text{there exists a } y\in Y \text{ such that } (x,y)\in R \text{ and } (y,z)\in S\}$$

together with the inclusion map j'. Thus the categorical definition of composition (1.26) yields in these special concrete categories the usual set theoretic composition.

A similar remark can be made about the definition of the inverse relation. That is, the categorical definition yields the usual set theoretic definition in the categories Set, Ab, Grp, and Top_1 to only mention a few. Indeed, the categorical definitions were obtained by analyzing the situation in the set theoretic case.

However, in the category Top_2 of Hausdorff spaces and continuous maps the extremal monomorphisms are the closed embeddings which leads to the following consequences.

1.34. Example. If (R,j) is a relation from X to Y and (S,k) is a relation from Y to Z for Top_2-objects X,Y, and Z. Let T be the following set.

$$\{(x,z)\colon \text{there exists a } y\in Y \text{ such that } (x,y)\in R \text{ and } (y,z)\in S\}$$

Then $(R\circ S,j') \equiv (\text{cl}T,j)$ where "cl" means closure with respect to the top-

ology of $X \times Z$ (c.f. 1.27).

Proof. Recall that the extremal monomorphisms are the closed embeddings, thus $R \circ S$ is a closed subset of $X \times Z$. Clearly the following diagram commutes.

$$\begin{tikzcd}
(R\times Z)\cap(X\times S) \arrow[r, "\gamma"] \arrow[dr] \arrow[ddr] & X\times Y\times Z \arrow[r, "\langle\bar{\pi}_1,\bar{\pi}_3\rangle"] & X\times Z \\
 & clT \arrow[ur, "\hat{j}"] & \\
 & R\circ S \arrow[uur, "j'"'] &
\end{tikzcd}$$

It is evident that $T \subseteq R \circ S$ whence $clT \subseteq R \circ S$. But $(R \circ S, j')$ is the intersection of all closed subsets of $X \times Z$ through which $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma$ factors. Thus $(R \circ S, j') \subset (clT, \hat{j})$. Hence $(R \circ S, j') \equiv (clT, \hat{j})$.

1.35. Example. With the hypothesis of Example 1.34, T and clT do not necessarily coincide.

Proof. Let $X = Z$ be the closed unit interval with the usual subspace topology induced from the real line. Let Y be the closed unit interval with the discrete topology. Let $R = \{(x,y): y = x\}$ considered as a subspace of $X \times Y$. Let $S = \{(y,z): 0 < y < \frac{1}{2}\}$ considered as a subspace of $Y \times Z$. It is easy to see that both R and S are closed in $X \times Y$ and $Y \times Z$ respectively.

Clearly $T = \{(x,z): 0 < x < \frac{1}{2}\}$ and $clT = \{(x,z): 0 \leq x \leq \frac{1}{2}\}$ whence $T \subsetneq clT$.

1.36. Example. In the category $\underline{\mathrm{Top}}_2$ the composition of relations is not necessarily associative.

Proof. Let $X = Z$ be the closed unit interval with the usual subspace topology induced from the real line. Let Y be the closed unit interval with the discrete topology. Let $R = \{(\frac{1}{2},\frac{1}{2})\}$ considered as a subspace of $X\times X$. Let S be $\{(x,y): y = x\}$ considered as a subspace of $X\times Y$ and let T be $\{(y,z): 0 < y < \frac{1}{2}\}$ where T is considered to be a subspace of $Y\times Z$. Hence, each together with its inclusion map is a relation since each of R, S, and T is a closed subspace of $X\times X$, $X\times Y$, and $Y\times Z$ respectively.

It follows that $R\circ S = \{(\frac{1}{2},\frac{1}{2})\}$ and that $(R\circ S)\circ T = \emptyset$. But $S\circ T = \{(y,z): 0 \leq y \leq \frac{1}{2}\}$ and from this it follows that $R\circ(S\circ T) = \{(\frac{1}{2},z): z\in Z\}$. Hence $R\circ(S\circ T) \neq (R\circ S)\circ T$.

1.37. Remark. At first glance, the results of Examples 1.34, 1.35 and 1.36 seem to be pathological, thereby casting doubt on the usefulness of the categorical definition of composition of relations (1.26). However, this should cause no more anxiety than does the fact that the set theoretic union of two subgroups of a group is seldom a subgroup.

Furthermore, the results 1.31, 1.38, 1.39, 2.4, 3.1, 3.6, 3.9, 3.10, 3.12, 4.22, 5.20, 5.23, 5.25, 5.26, 5.27, 5.34, 6.13 and 6.27 seem to indicate that this definition (1.26) yields nice theorems which re-enforces its appropriateness.

1.38. Theorem. Let (R,j) be a relation from X to Y and let (S,k) be a relation from Y to Z. Then $(R\circ S)^{-1}$ and $S^{-1}\circ R^{-1}$ are isomorphic relations from Z to X.

Proof. The following products shall be used:

$(X\times Y,\pi_1,\pi_2)$, $(Y\times Z,\tilde{\pi}_1,\tilde{\pi}_2)$, $(R\times Z,\rho_1{}^*,\rho_2{}^*)$, $(X\times Z,\rho_1,\rho_2)$, $(X\times S,\tilde{\rho}_1,\tilde{\rho}_2)$,

$(X\times(Y\times Z),\pi_1{*},\pi_2{*})$, $(X\times Y\times Z,\bar\pi_1,\bar\pi_2,\bar\pi_3)$, and $((X\times Y)\times Z,\bar\rho_1,\bar\rho_2)$.

The notation "^" over a projection morphism shall denote the projection morphism of that product object where the product is taken in reverse order; i.e., the projections of $Y\times X$ are $\hat\pi_1$ and $\hat\pi_2$ and the projections of $Z\times Y$ are $\hat{\hat\pi}_1$ and $\hat{\hat\pi}_2$.

Consider the following diagram. It will be shown to be commutative.

$$\begin{tikzcd}
 & & Z\times R^{-1} \arrow[dr, "1_Z\times j^*"] & \\
 & R\times Z \arrow[ur, "\langle\rho_2{*},\tau\rho_1{*}\rangle"] \arrow[r, "j\times 1_Z"] & (X\times Y)\times Z \arrow[r, "\langle\bar\rho_2{,}\langle\pi_2{,}\pi_1\rangle\bar\rho_1\rangle"] \arrow[d, "\Theta_1"] & Z\times(Y\times X) \arrow[d, "\tilde\Theta_1"] \\
(R\times Z)\cap(X\times S) \arrow[ur, "\lambda_1"] \arrow[rr, "\gamma"] \arrow[dr, "\lambda_2"'] & & X\times Y\times Z \arrow[r, "\langle\bar\pi_1{,}\bar\pi_2{,}\bar\pi_3\rangle"] & Z\times Y\times X \\
 & X\times S \arrow[r, "1_X\times k"] \arrow[dr, "\langle\tilde\tau\tilde\rho_2{,}\tilde\rho_1\rangle"'] & X\times(Y\times Z) \arrow[u, "\Theta_2"'] \arrow[r, "\langle\langle\tilde\pi_2{,}\tilde\pi_1\rangle\pi_2{*}{,}\pi_1{*}\rangle"] & (Z\times Y)\times X \arrow[u, "\tilde\Theta_2"'] \\
 & & S^{-1}\times X \arrow[ur, "k^*\times 1_X"'] &
\end{tikzcd}$$

$$\tilde{\Theta}_1 = <\hat{\bar{\rho}}_1, \hat{\pi}_1\hat{\bar{\rho}}_2, \hat{\pi}_2\hat{\bar{\rho}}_2> \text{ and } \tilde{\Theta}_2 = <\hat{\bar{\tilde{\pi}}}_1\hat{\pi}_1{}^*, \hat{\bar{\tilde{\pi}}}_2\hat{\pi}_1{}^*, \hat{\pi}_2{}^*>.$$

$\hat{\bar{\pi}}_1\tilde{\Theta}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \hat{\bar{\rho}}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \bar{\rho}_2.$

$\hat{\bar{\pi}}_1<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_1 = \bar{\pi}_3\Theta_1 = \bar{\rho}_2.$

$\hat{\bar{\pi}}_2\tilde{\Theta}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \hat{\pi}_1\hat{\bar{\rho}}_2<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \hat{\pi}_1<\pi_2, \pi_1>\bar{\rho}_1 = \pi_2\bar{\rho}_1.$

$\hat{\bar{\pi}}_2<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_1 = \bar{\pi}_2\Theta_1 = \pi_2\bar{\rho}_1.$

$\hat{\bar{\pi}}_3\tilde{\Theta}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \hat{\pi}_2\hat{\bar{\rho}}_2<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = \hat{\pi}_2<\pi_2, \pi_1>\bar{\rho}_1 = \pi_1\bar{\rho}_1.$

$\hat{\bar{\pi}}_3<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_1 = \bar{\pi}_1\Theta_1 = \pi_1\bar{\rho}_1.$

$\hat{\bar{\pi}}_1\tilde{\Theta}_2<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \hat{\tilde{\pi}}_1\hat{\pi}_1{}^*<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \hat{\tilde{\pi}}_1<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^* = \tilde{\pi}_2\pi_2{}^*.$

$\hat{\bar{\pi}}_1<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_2 = \bar{\pi}_3\Theta_2 = \tilde{\pi}_2\pi_2{}^*.$

$\hat{\bar{\pi}}_2\tilde{\Theta}_2<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \hat{\tilde{\pi}}_2\hat{\pi}_1{}^*<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \hat{\tilde{\pi}}_2<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^* = \tilde{\pi}_1\pi_2{}^*.$

$\hat{\bar{\pi}}_2<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_2 = \bar{\pi}_2\Theta_2 = \tilde{\pi}_1\pi_2{}^*.$

$\hat{\bar{\pi}}_3\tilde{\Theta}_2<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \hat{\pi}_2{}^*<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*> = \pi_1{}^*.$

$\hat{\bar{\pi}}_3<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_2 = \bar{\pi}_1\Theta_2 = \pi_1{}^*.$

Thus $\tilde{\Theta}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1> = <\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_1$ and $<\bar{\pi}_3, \bar{\pi}_2, \bar{\pi}_1>\Theta_2 = \tilde{\Theta}_2<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*>.$

$\ddot{\bar{\rho}}_1<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1>(j\times 1) = \bar{\rho}_2(j\times 1) = \rho_2{}^*.$

$\hat{\bar{\rho}}_1(1\times j^*)<\rho_2{}^*, \tau\rho_1{}^*> = \hat{\rho}_1{}^*<\rho_2{}^*, \tau\rho_1{}^*> = \rho_2{}^*.$

$\hat{\bar{\rho}}_2<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1>(j\times 1) = <\pi_2, \pi_1>\bar{\rho}_1(j\times 1) = <\pi_2, \pi_1>j\rho_1{}^* = j^*\tau\rho_1{}^*.$

$\hat{\bar{\rho}}_2(1\times j^*)<\rho_2{}^*, \tau\rho_1{}^*> = j^*\hat{\rho}_2{}^*<\rho_2{}^*, \tau\rho_1{}^*> = j^*\tau\rho_1{}^*.$

Thus $<\bar{\rho}_2, <\pi_2, \pi_1>\bar{\rho}_1>(j\times 1) = (1\times j^*)<\rho_2{}^*, \tau\rho_1{}^*>.$

$\hat{\tilde{\pi}}_1<<\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*, \pi_1{}^*>(1\times k) = <\tilde{\pi}_2, \tilde{\pi}_1>\pi_2{}^*(1\times k) = <\tilde{\pi}_2, \tilde{\pi}_1>k\tilde{\rho}_2 = k^*\tilde{\tau}\tilde{\rho}_2.$

$\hat{\tilde{\pi}}_1(k^*\times 1)<\tilde{\tau}\tilde{\rho}_2, \tilde{\rho}_1> = k^*\hat{\tilde{\rho}}_1<\tilde{\tau}\tilde{\rho}_2, \tilde{\rho}_1> = k^*\tilde{\tau}\tilde{\rho}_2.$

$\hat{\tilde{\pi}}_2<<\tilde{\pi}_2,\tilde{\pi}_1>\pi_2^*,\pi_1^*>(1\times k) = \pi_1^*(1\times k) = \tilde{\rho}_1.$

$\hat{\tilde{\pi}}_2(k^*\times 1)<\tilde{\tau}\tilde{\rho}_2,\tilde{\rho}_1> = \hat{\tilde{\rho}}_2<\tilde{\tau}\tilde{\rho}_2,\tilde{\rho}_1> = \tilde{\rho}_1.$

Thus $<<\tilde{\pi}_2,\tilde{\pi}_1>\pi_2^*,\pi_1^*>(1\times k) = (k^*\times 1)<\tilde{\tau}\tilde{\rho}_2,\tilde{\rho}_1>$.

Hence the diagram is commutative.

Consequently by the definition of intersection, there exists a unique morphism ξ such that the following diagram commutes.

$$\begin{tikzcd}
(R\times Z)\cap(X\times S) \arrow[r, tail, "\gamma"] \arrow[d, "\xi"'] & X\times Y\times Z \arrow[d, "<\bar{\pi}_3,\bar{\pi}_2,\bar{\pi}_1>"] \\
(S^{-1}\times X)\cap(Z\times R^{-1}) \arrow[r, tail, "\delta"'] & Z\times Y\times X
\end{tikzcd}$$

It is easy to see that the following diagram commutes.

$$\begin{tikzcd}
(R\times Z)\cap(X\times S) \arrow[r, tail, "\gamma"] \arrow[d, "\xi"'] & X\times Y\times Z \arrow[r, "<\bar{\pi}_1,\bar{\pi}_3>"] & X\times Z \\
(S^{-1}\times X)\cap(Z\times R^{-1}) \arrow[rr, "<\hat{\bar{\pi}}_1\ \hat{\bar{\pi}}_3>\delta"] \arrow[dr, "\tau'"'] & & Z\times X \arrow[u, "<\hat{\rho}_2\ \hat{\rho}_1>"'] \\
& S^{-1}oR^{-1} \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

Since (RoS,β) is the intersection of all extremal subobjects through which $<\pi_1,\pi_3>\gamma$ factors, then $(RoS,\beta) \leq (S^{-1}oR^{-1},<\hat{\rho}_2,\hat{\rho}_1>\alpha')$. Hence there is some morphism μ such that $<\hat{\rho}_2,\hat{\rho}_1>\alpha'\mu = \beta$. Consequently, $<\rho_2,\rho_1><\hat{\rho}_2,\hat{\rho}_1>\alpha'\mu = <\rho_2,\rho_1><\rho_2,\rho_1>^{-1}\alpha'\mu = \alpha'\mu$ (1.6). Thus $<\rho_2,\rho_1>\delta = \alpha'\mu = \beta^*\tau^*$ and the following diagram commutes.

$$\begin{tikzcd}
 & S^{-1}oR^{-1} \arrow[dr, "\langle\hat{\rho}_2,\hat{\rho}_1\rangle\alpha'"] & & \\
RoS \arrow[ur, "\mu"] \arrow[rr, "\beta"] \arrow[dr, "\tau^*"'] & & X\times Z \arrow[r, "\langle\rho_2,\rho_1\rangle"] & Z\times X \\
 & (RoS)^{-1} \arrow[urr, "\beta^*"'] & &
\end{tikzcd}$$

Since $((RoS)^{-1},\beta^*)$ is the intersection of all extremal subobjects through which $\langle\rho_2,\rho_1\rangle\beta$ factors then $((RoS)^{-1},\beta^*) \leq (S^{-1}oR^{-1},\alpha')$.

Now applying the above result to (S^{-1},k^*) and (R^{-1},j^*), it follows that $((S^{-1}oR^{-1})^{-1},\alpha'^*) \leq ((R^{-1})^{-1}o(S^{-1})^{-1},\beta\#) \equiv (RoS,\beta)$ (1.11) whence $(S^{-1}oR^{-1},\alpha') \leq ((RoS)^{-1},\beta^*)$ (1.12), so that

$$(S^{-1}oR^{-1},\alpha') \equiv ((RoS)^{-1},\beta^*).$$

1.39. Corollary. Let (R,j) be a relation from X to Y. Then $(RoR^{-1},j\#)$ is a symmetric relation on X and $(R^{-1}oR,j')$ is symmetric on Y.

Proof. $((RoR^{-1})^{-1},j\#^*) \equiv ((R^{-1})^{-1}oR^{-1},j) \equiv (RoR^{-1},j\#)$ and $((R^{-1}oR)^{-1},j'^*) \equiv (R^{-1}o(R^{-1})^{-1},j) \equiv (R^{-1}oR,j')$ (1.38 and 1.11).

1.40. Proposition. Let (R,j) be a relation from X to Y and let (S,k) and (T,m) be relations from Y to Z. Then

$$(Ro(S\cap T),\beta) \leq ((RoS)\cap(RoT),\delta).$$

Proof. By Proposition 1.5 there exist canonical isomorphisms:

$$\psi:\ (R\times Z)\cap(X\times(S\cap T)) \twoheadrightarrow (R\times Z)\cap(X\times S)\cap(X\times T)$$

$$\bar{\psi}:\ X\times(S\cap T) \twoheadrightarrow (X\times S)\cap(X\times T).$$

Consider the following commutative diagrams.

$$\begin{tikzcd}
S\cap T \arrow[r, "\lambda_2"] \arrow[d, "\lambda_1"'] \arrow[dr, "\alpha"] & S \arrow[d, "k"] \\
T \arrow[r, "m"'] & Y\times Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & R\times Z \arrow[rr, "j\times 1"] & & (X\times Y)\times Z \arrow[dr, "\theta_1"] & \\
(R\times Z)\cap(X\times(S\cap T)) \arrow[ur, "\lambda_3"] \arrow[rr, "\psi"] \arrow[dr, "\lambda_4"'] & & (R\times Z)\cap(X\times S)\cap(X\times T) \arrow[rr, "\gamma"] \arrow[d, "\lambda_5"] & & X\times Y\times Z \\
 & X\times(S\cap T) \arrow[r, "\bar{\psi}"] \arrow[drr, "1\times\lambda_1"] \arrow[ddr, "1\times\lambda_2"'] & (X\times S)\cap(X\times T) \arrow[rr, "\bar{\beta}"] \arrow[dr, "\lambda_7"] \arrow[dd, "\lambda_6"] & & X\times(Y\times Z) \arrow[u, "\theta_2"'] \\
 & & & X\times T \arrow[ur, "1\times m"'] & \\
 & & X\times S \arrow[uurr, "1\times k"'] & &
\end{tikzcd}$$

Note that $\bar{\beta}\bar{\psi} = (1\times k)(1\times\lambda_2) = (1\times m)(1\times\lambda_1) = 1\times\alpha$.

Let (τ,β) be the epi-extremal mono factorization of $<\bar{\pi}_1,\bar{\pi}_3>\gamma\psi$. Thus the codomain of τ (domain of β) is $Ro(S\cap T)$. Since this is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>\gamma\psi$ factors it follows that $(Ro(S\cap T),\beta) \leq (RoS,\delta_1)$ and $(Ro(S\cap T),\beta) \leq (RoT,\delta_2)$. Thus

$$(Ro(S\cap T),\beta) \leq ((RoS)\cap(RoT),\delta) \quad (1.19).$$

SECTION 2. CATEGORICAL CONGRUENCES

2.0. Remark. Lambek [16,pg 9] presents the following definitions for dealing with rings which have identities.

> More general than homomorphism is the concept of homomorphic relation. Thus let θ be a binary relation between rings R and S, that is essentially a subset of the Cartesian product R×S, then θ is called homomorphic if $0\theta 0$, $1\theta 1$, and $r_1\theta s_1$, $r_2\theta s_2$ imply $(-r_1)\theta(-s_1)$, $(r_1+r_2)\theta(s_1+s_2)$, $(r_1r_2)\theta(s_1s_2)$. Of course a similar definition can be made for any equationally defined class of algebraic systems.

He goes on to add:

> A homomorphic relation on R (that is, between R and itself) is called a congruence relation if it is an equivalence relation, that is reflexive, symmetric, and transitive.

Lambek notes that a symmetric transitive relation is not necessarily reflexive, but is a congruence on a subring. He also notes that a reflexive homomorphic relation is a congruence. This latter result is due to the fact that all homomorphic relations are difunctional (see 5.22).

We will generalize all of these results. However it must be noted that in the category Rng[1] a congruence is an equivalence relation and conversely. Thus we shall obtain the result that if (R,j) is a symmetric transitive relation on an object X then R is an equivalence relation on an extremal subobject of X. However, this result must be postponed until Section 3 (see 3.4 and 3.10).

Also the result that the reflexive difunctional relations are precisely the equivalence relations must be postponed until Section 5.

In this section it will be shown that a (categorical) congruence

is a (categorical) equivalence relation and that congruences (when $\mathfrak{C}$ has coproducts) are determined by (categorical) quotients (2.12).

If f is a set function from a set X to a set Y then the set

$$\{(x_1,x_2)\varepsilon X\times X: f(x_1) = f(x_2)\}$$

is called the congruence (sometimes kernel) determined by f. It will be shown that (categorical) congruences have behavior similar to that of the above set (2.8, 2.10, 2.11, and 2.12).

2.1. Definition. If (R,j) is a subobject of $X\times X$ then (R,j) is called a congruence if and only if there exists a morphism f with domain X such that (R,j) is the equalizer of $f\pi_1$ and $f\pi_2$.

$$R \rightarrowtail{j} \xrightarrow{j} X\times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

If g is a morphism with domain X then the equalizer of $g\pi_1$ and $g\pi_2$ denoted by $(\mathrm{cong}(g),i_g)$ is called the congruence generated by g.

2.2. Remark. If X is a $\mathfrak{C}$-object then (Δ_X,i_X) is the congruence generated by 1_X.

2.3. Remark. It is easy to see that (R,j) is a congruence on X if and only if (R^{-1},j^*) is a congruence on X.

2.4. Theorem. If (R,j) is a congruence on X then (R,j) is an equivalence relation on X.

Proof. Since (R,j) is a congruence on X there exists a morphism f with domain X such that (R,j) is the equalizer of $f\pi_1$ and $f\pi_2$. Recall that (Δ_X,i_X) is the congruence generated by 1_X whence $1_X\pi_1 i_X = 1_X\pi_2 i_X$. Thus $f\pi_1 i_X = f\pi_2 i_X$ so by the definition of equalizer there exists a morphism

λ from Δ_X to R for which $j\lambda = i_X$. This implies that $(\Delta_X, i_X) \leq (R,j)$ so that (R,j) is reflexive.

To see that (R,j) is symmetric, observe that $f\pi_1\langle\pi_2,\pi_1\rangle j = f\pi_2 j = f\pi_1 j = f\pi_2\langle\pi_2,\pi_1\rangle j$. Thus $f\pi_1 j^*\tau = f\pi_2 j^*\tau$ so that since τ is an epimorphism it follows that $f\pi_1 j^* = f\pi_2 j^*$. Hence, from the definition of equalizer, there exists a morphism η from R^{-1} to R for which $j\eta = j^*$. This implies that $(R^{-1}, j^*) \leq (R,j)$ so that (R,j) is symmetric.

Consider the following products: $(X\times X, \pi_1, \pi_2)$, $(X\times X\times X, \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$, $((X\times X)\times X, \rho_1, \rho_2)$, $(X\times(X\times X), \bar{\rho}_1, \bar{\rho}_2)$, $(R\times X, \pi_1^*, \pi_2^*)$ and $(X\times R, \tilde{\pi}_1, \tilde{\pi}_2)$. To see that (R,j) is transitive, consider the following commutative diagram.

$$\begin{tikzcd}
 & R\times X \arrow[r, tail, "j\times 1_X"] & (X\times X)\times X \arrow[dr, "\theta_1"] & \\
(R\times X)\cap(X\times R) \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] \arrow[rrr, tail, "\gamma"] & & & X\times X\times X \\
 & X\times R \arrow[r, tail, "1_X\times j"'] & X\times(X\times X) \arrow[ur, "\theta_2"'] &
\end{tikzcd}$$

Let $(\tau\#, j\#)$ be the epi-extremal mono factorization of $\langle\bar{\pi}_1, \bar{\pi}_3\rangle\gamma$. Recall that the codomain of $\tau\#$ (domain of $j\#$) is $R\circ R$.

Next, it will be shown that $f\bar{\pi}_1\gamma = f\bar{\pi}_2\gamma = f\bar{\pi}_3\gamma$.

$$f\bar{\pi}_3\gamma = f\bar{\pi}_3\theta_2(1_X\times j)\lambda_2 = f\pi_2\bar{\rho}_2(1_X\times j)\lambda_2 = f\pi_2 j\tilde{\pi}_2\lambda_2 = f\pi_1 j\tilde{\pi}_2\lambda_2 =$$
$$f\pi_1\bar{\rho}_2(1_X\times j)\lambda_2 = f\bar{\pi}_2\theta_2(1_X\times j)\lambda_2 = f\bar{\pi}_2\gamma.$$
$$f\bar{\pi}_1\gamma = f\bar{\pi}_1\theta_1(j\times 1_X)\lambda_1 = f\pi_1\rho_1(j\times 1_X)\lambda_1 = f\pi_1 j\pi_1^*\lambda_1 = f\pi_2 j\pi_1^*\lambda_1 =$$
$$f\pi_2\rho_1(j\times 1_X)\lambda_1 = f\bar{\pi}_2\theta_1(j\times 1_X)\lambda_1 = f\bar{\pi}_2\gamma.$$

Thus $f\pi_1\langle\bar{\pi}_1, \bar{\pi}_3\rangle\gamma = f\bar{\pi}_1\gamma = f\bar{\pi}_3\gamma = f\pi_2\langle\bar{\pi}_1, \bar{\pi}_3\rangle\gamma$; so $f\pi_1 j\#\tau\# = f\pi_2 j\#\tau\#$.

Again, since $\tau\#$ is an epimorphism, it follows that $f\pi_1 j\# = f\pi_2 j\#$. By the definition of equalizer there exists a morphism k from RoR to R for which $jk = j\#$. This implies that $(RoR, j\#) \leq (R,j)$ so that (R,j) is transitive.

2.5. Theorem. The intersection of any finite family of congruences on any $\mathcal{C}$-object is a congruence.

Proof. Let $\{(E_i,e_i): i\varepsilon I\}$ be a finite family of congruences on X. Then there exist morphisms f_i with domain X such that (E_i,e_i) is the equalizer of $f_i\pi_1$ and $f_i\pi_2$ (2.1). Let the codomain of each f_i be denoted Y_i.

Consider the morphism $\underset{i\varepsilon I}{<f_i>}$ from X to $\prod_{i\varepsilon I} Y_i$ and consider the intersection $(\bigcap_{i\varepsilon I} E_i, e)$.

It will be shown that $(\bigcap_{i\varepsilon I} E_i, e)$ is the equalizer of $\underset{i\varepsilon I}{<f_i>}\pi_1$ and $\underset{i\varepsilon I}{<f_i>}\pi_2$.

$$\begin{tikzcd}
\bigcap_{i\varepsilon I} E_i \arrow[r, "e"] \arrow[d, "\lambda_i"'] & X\times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[rr, "f_i"] \arrow[dr, "\underset{i\varepsilon I}{<f_i>}"'] & & Y_i \\
E_i \arrow[ur, "e_i"'] & & & \prod_{i\varepsilon I} Y_i \arrow[ur, "\rho_i"'] &
\end{tikzcd}$$

First observe that: $\rho_j \underset{i\varepsilon I}{<f_i>}\pi_1 e = f_j\pi_1 e = f_j\pi_2 e = \rho_j \underset{i\varepsilon I}{<f_i>}\pi_2 e$ for each $j\varepsilon I$. Thus $\underset{i\varepsilon I}{<f_i>}\pi_1 e = \underset{i\varepsilon I}{<f_i>}\pi_2 e$.

Now if g is a morphism from W to X×X such that $\underset{i\varepsilon I}{<f_i>}\pi_1 g = \underset{i\varepsilon I}{<f_i>}\pi_2 g$ then $f_j\pi_1 g = \rho_j \underset{i\varepsilon I}{<f_i>}\pi_1 g = \rho_j \underset{i\varepsilon I}{<f_i>}\pi_2 g = f_j\pi_2 g$. Thus by the definition of equalizer there exist morphisms k_i from W to E_i so that $e_i k_i = g$ for each $i\varepsilon I$. Thus by the definition of intersection there exists a morphism

k from W to $\bigcap_{i\varepsilon I} E_i$ such that $ek = g$. This implies that $(\bigcap_{i\varepsilon I} E_i, e)$ is the equalizer of $\langle f_i\rangle_{i\varepsilon I}\pi_1$ and $\langle f_i\rangle_{i\varepsilon I}\pi_2$.

2.6. Proposition. If $\mathfrak{C}$ is complete then the intersection of any family of congruences on any $\mathfrak{C}$-object is a congruence.

Proof. Repeat the proof of 2.7 assuming I to be infinite.

2.7. Proposition. Let Ω be the family of all congruences on X and let $(\bigcap\Omega, \rho)$ be the intersection of this family. Then $\bigcap\Omega$ and Δ_X are isomorphic relations on X.

Proof. If $(E,e)\varepsilon\Omega$ then (E,e) is an equivalence relation and hence is reflexive (2.4). Thus $(\Delta_X, i_X) \leq (E,e)$. Hence $(\Delta_X, i_X) \leq (\bigcap\Omega, \rho)$. But (Δ_X, i_X) is a congruence; hence $(\bigcap\Omega, \rho) \leq (\Delta_X, i_X)$.

2.8. Proposition. Let f be a $\mathfrak{C}$-morphism from X to Y. Then f is a monomorphism if and only if Δ_X and cong(f) are isomorphic relations on X.

Proof. Since $(\text{cong}(f), i_f)$ is an equivalence relation (2.4) it is reflexive and hence $(\Delta_X, i_X) \leq (\text{cong}(f), i_f)$. If f is a monomorphism then $f\pi_1 i_f = f\pi_2 i_f$ implies that $\pi_1 i_f = \pi_2 i_f$. Hence there exists a morphism k for which $i_X k = i_f$ and consequently $(\text{cong}(f), i_f) \leq (\Delta_X, i_X)$.

Conversely, suppose that $(\text{cong}(f), i_f) \equiv (\Delta_X, i_X)$ and α and β are morphisms having domain Z and codomain X such that $f\alpha = f\beta$. Consider the morphism $\langle\alpha,\beta\rangle$ from Z to $X\times X$. $f\pi_1\langle\alpha,\beta\rangle = f\alpha = f\beta = f\pi_2\langle\alpha,\beta\rangle$ so that there exists a morphism λ from Z to Δ_X for which $i_X\lambda = \langle\alpha,\beta\rangle$. Thus $\alpha = \pi_1\langle\alpha,\beta\rangle = \pi_1 i_X\lambda = \pi_2 i_X\lambda = \pi_2\langle\alpha,\beta\rangle = \beta$. Consequently $\alpha = \beta$ so that f is a monomorphism.

2.9. Definition. A $\mathfrak{C}$-morphism f from X to Y is said to be constant if and only if for all pairs of morphisms $Z \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$, $f\alpha = f\beta$.

2.10. <u>Proposition</u>. Let f be a morphism from X to Y. Then f is constant if and only if $(\mathrm{cong}(f), i_f) \equiv (X\times X, 1_{X\times X})$.

<u>Proof</u>. If f is constant then $f\pi_1 = f\pi_2$ so that $f\pi_1 1_{X\times X} = f\pi_2 1_{X\times X}$. Thus there exists a unique morphism k from $X\times X$ to $\mathrm{cong}(f)$ for which $i_f k = 1_{X\times X}$ whence i_f is a retraction. But since i_f is an equalizer, it must be an isomorphism (0. 4) so that $(\mathrm{cong}(f), i_f)$ and $(X\times X, 1_{X\times X})$ are isomorphic relations on X.

Conversely, suppose that $(X\times X, 1_{X\times X}) \equiv (\mathrm{cong}(f), i_f)$ and that α and β are morphisms with common domain, and codomain X. Consider $\langle\alpha,\beta\rangle$ from Z to $X\times X$ where Z is the common domain of α and β. Since $f\pi_1 1_{X\times X} = f\pi_2 1_{X\times X}$, it follows that $f\pi_1 = f\pi_2$ so that $f\alpha = f\pi_1\langle\alpha,\beta\rangle = f\pi_2\langle\alpha,\beta\rangle = f\beta$. Thus f is a constant morphism.

2.11. <u>Proposition</u>. If f from X to Y, g from Z to Y, and h from X to Z are $\mathcal{C}$-morphisms such that $f = gh$ then $(\mathrm{cong}(h), i_h) \leq (\mathrm{cong}(f), i_f)$. Furthermore if g is a monomorphism then $(\mathrm{cong}(h), i_h) \equiv (\mathrm{cong}(f), i_f)$.

<u>Proof</u>. Since $h\pi_1 i_h = h\pi_2 i_h$ it follows that $gh\pi_1 i_h = gh\pi_2 i_h$ so that $f\pi_1 i_h = f\pi_2 i_h$. Thus there exists a morphism k from $\mathrm{cong}(h)$ to $\mathrm{cong}(f)$ for which $i_f k = i_h$. Whence $(\mathrm{cong}(h), i_h) \leq (\mathrm{cong}(f), i_f)$.

If g is a monomorphism then $f\pi_1 i_f = f\pi_2 i_f = gh\pi_1 i_f = gh\pi_2 i_f$ implies that $h\pi_1 i_f = h\pi_2 i_f$. Thus there exists a morphism k^* from $\mathrm{cong}(f)$ to $\mathrm{cong}(h)$ for which $i_h k^* = i_f$, whence $(\mathrm{cong}(f), i_f) \leq (\mathrm{cong}(h), i_h)$. Consequently $(\mathrm{cong}(f), i_f) \equiv (\mathrm{cong}(h), i_h)$.

2.12. Proposition. If $\mathcal{C}$ has coequalizers and f is a $\mathcal{C}$-morphism from X to Y and if (f^*, Z) is the coequalizer of $\pi_1 i_f$ and $\pi_2 i_f$ then $\mathrm{cong}(f)$ and $\mathrm{cong}(f^*)$ are isomorphic relations on X.

<u>Proof</u>. Since $f\pi_1 i_f = f\pi_2 i_f$ then by the definition of coequalizer there

exists a morphism k^* from Z to Y for which $k^*f^* = f$. Since $f^*\pi_1 i_{f^*} = f^*\pi_2 i_{f^*}$ it follows that $f\pi_1 i_{f^*} = k^*f^*\pi_1 i_{f^*} = k^*f^*\pi_2 i_{f^*} = f\pi_2 i_{f^*}$. Thus there exists a morphism k from cong(f^*) to cong(f) for which $i_f k = i_{f^*}$. Consequently $(\mathrm{cong}(f^*), i_{f^*}) \leq (\mathrm{cong}(f), i_f)$.

Now since f^* is the coequalizer of $\pi_1 i_f$ and $\pi_2 i_f$ then $f^*\pi_1 i_f = f^*\pi_2 i_f$. Hence there exists a morphism k' from cong(f) to cong(f^*) for which $i_{f^*}k' = i_f$. Consequently $(\mathrm{cong}(f), i_f) \leq (\mathrm{cong}(f^*), i_{f^*})$.

2.13. <u>Proposition</u>. If $\mathcal{E}$ is complete and Ω is a family of congruences on X generated by morphisms $f\colon X \longrightarrow Y_f$ and if $(\bigcap\Omega, \rho) \equiv (\Delta_X, i_X)$ then the unique morphism Θ from X to $\prod Y_f$ such that $\pi_f\Theta = f$, is a monomorphism.

<u>Proof</u>. Observe that for each f, $f\pi_1 i_\Theta = \pi_f\Theta\pi_1 i_\Theta = \pi_f\Theta\pi_2 i_\Theta = f\pi_2 i_\Theta$. Thus it follows that $(\mathrm{cong}(\Theta), i_\Theta) \leq (\mathrm{cong}(f), i_f)$ for all $X \xrightarrow{f} Y_f$. Hence $(\mathrm{cong}(\Theta), i_\Theta) \leq (\bigcap\Omega, \rho)$ (1.19). Since $(\bigcap\Omega, \rho) \equiv (\Delta_X, i_X) \leq (\mathrm{cong}(\Theta), i_\Theta)$ (2.4) it follows that $(\bigcap\Omega, \rho) \equiv (\Delta_X, i_X) \equiv (\mathrm{cong}(\Theta), i_\Theta)$. Thus Θ is a monomorphism (2.8).

2.14. <u>Corollary</u>. If $\mathcal{E}$ is complete and Ω is a family of congruences on X generated by morphisms $f\colon X \longrightarrow Y_f$ and for some $g\colon X \longrightarrow Y_g$, g is a monomorphism, then the unique morphism Θ from X to $\prod Y_f$ such that $\pi_f\Theta = f$ is a monomorphism.

<u>Proof</u>. Since g is a monomorphism then $(\mathrm{cong}(g), i_g) \equiv (\Delta_X, i_X)$ (2.8). Thus $(\bigcap\Omega, \rho) \leq (\Delta_X, i_X)$ by the definition of intersection. But $(\Delta_X, i_X) \leq (\bigcap\Omega, \rho)$ (2.4 and 2.6). Consequently $(\Delta_X, i_X) \equiv (\bigcap\Omega, \rho)$ and the result follows from Proposition 2.13.

SECTION 3. CATEGORICAL EQUIVALENCE RELATIONS AND QUASI-EQUIVALENCE RELATIONS

3.1. Theorem. If $\{(E_i,\phi_i): i\varepsilon I\}$ is a family of equivalence relations on a $\mathcal{B}$-object X then their intersection $(\bigcap_{i\varepsilon I} E_i,\phi)$ is an equivalence relation on X.

Proof. Since $(\Delta_X,i_X) \leq (E_i,\phi_i)$ for each $i\varepsilon I$ it follows that $(\Delta_X,i_X) \leq (\bigcap_{i\varepsilon I} E_i,\phi)$ (1.19). Hence $(\bigcap_{i\varepsilon I} E_i,\phi)$ is reflexive.

Since $(\bigcap_{i\varepsilon I} E_i,\phi) \leq (E_i,\phi_i)$ for each $i\varepsilon I$ and since each (E_i,ϕ_i) is symmetric it follows that $((\bigcap_{i\varepsilon I} E_i)^{-1},\phi^*) \leq (E_i^{-1},\phi_i^*) \leq (E_i,\phi_i)$ for each $i\varepsilon I$ (1.12). Thus $((\bigcap_{i\varepsilon I} E_i)^{-1},\phi^*) \leq (\bigcap_{i\varepsilon I} E_i,\phi)$ and hence $(\bigcap_{i\varepsilon I} E_i,\phi)$ is symmetric.

Since $(\bigcap_{i\varepsilon I} E_i,\phi) \leq (E_i,\phi_i)$ for each $i\varepsilon I$ then $((\bigcap_{i\varepsilon I} E_i)\circ(\bigcap_{i\varepsilon I} E_i),\phi\#) \leq (E_i\circ E_i,\phi_i\#)$ for each $i\varepsilon I$ (1.30). And since $(E_i\circ E_i,\phi_i\#) \leq (E_i,\phi_i)$ for each $i\varepsilon I$ it follows that $((\bigcap_{i\varepsilon I} E_i)\circ(\bigcap_{i\varepsilon I} E_i),\phi\#) \leq (\bigcap_{i\varepsilon I} E_i,\phi)$ (1.19) whence $(\bigcap_{i\varepsilon I} E_i,\phi)$ is transitive. Thus it is an equivalence relation.

3.2. Definition. A quasi-equivalence (R,j) on X is a relation on X which is both symmetric and transitive.

This term is due to Riguet [22]; however, Lambek [13] calls symmetric transitive relations subcongruences. While this term subcongruence is appropriate in the categories Grp and Ab, it does not seem to be appropriate in more general categories. MacLane [18] calls such relations symmetric idempotents.

3.3. Proposition. If (A,a) is an extremal subobject of X then $(A\times A, a\times a)$ is a quasi-equivalence on X.

Proof. Consider the products $(A\times A, \rho_1, \rho_2)$ and $(X\times X, \pi_1, \pi_2)$. Since a is an extremal monomorphism then $a\times a$ is an extremal monomorphism (0.20) and hence $(A\times A, a\times a)$ is a relation on X.

Consider the following commutative diagram.

$$\begin{tikzcd}
A\times A \arrow[r, tail, "a\times a"] \arrow[dr, two heads, "\tau"'] & X\times X \arrow[r, tail, two heads, "<\pi_2,\pi_1>"] & X\times X \\
& (A\times A)^{-1} \arrow[ur, tail, "(a\times a)^*"'] &
\end{tikzcd}$$

Since $\pi_1<\pi_2,\pi_1>(a\times a) = \pi_2(a\times a) = a\rho_2$;

$\pi_1(a\times a)<\rho_2,\rho_1> = a\rho_1<\rho_2,\rho_1> = a\rho_2$;

$\pi_2<\pi_2,\pi_1>(a\times a) = \pi_1(a\times a) = a\rho_1$; and

$\pi_2(a\times a)<\rho_2,\rho_1> = a\rho_2<\rho_2,\rho_1> = a\rho_1$

then it follows that $<\pi_2,\pi_1>(a\times a) = (a\times a)<\rho_2,\rho_1>$. But $<\rho_2,\rho_1>$ is an isomorphism hence an epimorphism and $a\times a$ is an extremal monomorphism; thus by the uniqueness of the epi-extremal mono factorization of $<\pi_2,\pi_1>(a\times a)$ (0.18), $((A\times A)^{-1},(a\times a)^*) \equiv (A\times A, a\times a)$. Thus $(A\times A, a\times a)$ is symmetric.

To see that $(A\times A, a\times a)$ is transitive, first consider $(((A\times A)\times X)\cap(X\times(A\times A)),\gamma)$ where γ is the unique extremal monomorphism induced by the indicated intersection. It will next be shown that $(A\times A\times A, a\times a\times a)$ and $(((A\times A)\times X)\cap(X\times(A\times A)),\gamma)$ are isomorphic as extremal subobjects of $X\times X\times X$. To show this it will be shown that $(A\times A\times A, a\times a\times a)$ is precisely the intersection of $((A\times A)\times X, \theta_1((a\times a)\times 1_X))$ and $(X\times(A\times A), \theta_2(1_X\times(a\times a)))$.

Consider the products: $(X\times X\times X,\bar\pi_1,\bar\pi_2,\bar\pi_3)$, $(X\times(A\times A),\tilde\rho_1,\tilde\rho_2)$, $((A\times A)\times X,\hat\rho_1,\hat\rho_2)$, $(X\times(X\times X),\pi_1{}^*,\pi_2{}^*)$, $((X\times X)\times X,\hat\pi_1{}^*,\hat\pi_2{}^*)$ and $(A\times A\times A,\bar\rho_1,\bar\rho_2,\bar\rho_3)$.

Observe the following equalities.

$$\bar\pi_1\Theta_1((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \pi_1\hat\pi_1{}^*((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \pi_1(a\times a)\hat\rho_1<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \pi_1(a\times a)<\bar\rho_1,\bar\rho_2> = a\rho_1<\bar\rho_1,\bar\rho_2> = a\bar\rho_1 = \bar\pi_1(a\times a\times a).$$

$$\bar\pi_2\Theta_1((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \pi_2\hat\pi_1{}^*((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \pi_2(a\times a)<\bar\rho_1,\bar\rho_2> = a\rho_2<\rho_1,\rho_2> = a\bar\rho_2 = \bar\pi_2(a\times a\times a).$$

$$\bar\pi_3\Theta_1((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = \hat\pi_2{}^*((a\times a)\times 1_X)<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = 1_X\hat\rho_2<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3> = 1_Xa\bar\rho_3 = a\bar\rho_3 = \bar\pi_3(a\times a\times a).$$

$$\bar\pi_1\Theta_2(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_1{}^*(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = 1_X\tilde\rho_1<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = 1_Xa\bar\rho_1 = a\bar\rho_1 = \bar\pi_1(a\times a\times a).$$

$$\bar\pi_2\Theta_2(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_1\pi_2{}^*(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_1(a\times a)\tilde\rho_2<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_1(a\times a)<\bar\rho_2,\bar\rho_3> = a\rho_1<\bar\rho_2,\bar\rho_3> = a\bar\rho_2 = \bar\pi_2(a\times a\times a).$$

$$\bar\pi_3\Theta_2(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_2\pi_2{}^*(1_X\times(a\times a))<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>> = \pi_2(a\times a)<\bar\rho_2,\bar\rho_3> = a\rho_2<\bar\rho_2,\bar\rho_3> = a\bar\rho_3 = \bar\pi_3(a\times a\times a).$$

Thus by the definition of product the following diagram commutes.

$$\begin{tikzcd}
 & (A\times A)\times X \arrow[rr, "(a\times a)\times 1_X"] & & (X\times X)\times X \arrow[dr, "\theta_1"] & \\
A\times A\times A \arrow[ur, "<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3>"] \arrow[rrrr, "a\times a\times a"] \arrow[dr, "<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>>"'] & & & & X\times X\times X \\
 & X\times(A\times A) \arrow[rr, "1_X\times(a\times a)"'] & & X\times(X\times X) \arrow[ur, "\theta_2"'] &
\end{tikzcd}$$

Now, if (W,δ) is a subobject of $X\times X\times X$ so that there exist morphisms γ_1 and γ_2 such that $\theta_1((a\times a)\times 1_X)\gamma_1 = \delta = \theta_2(1_X\times(a\times a))\gamma_2$ then consider the morphism $<\rho_1\hat{\bar\rho}_1\gamma_1, \rho_2\hat{\bar\rho}_1\gamma_1, \rho_2\tilde\rho_2\gamma_2> = \xi$ from W to $A\times A\times A$. It will be shown that $<<\bar\rho_1,\bar\rho_2>,a\rho_3>\xi = \gamma_1$ and $<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>>\xi = \gamma_2$.

Since $\rho_1<\bar\rho_1,\bar\rho_2>\xi = \bar\rho_1\xi = \rho_1\hat{\bar\rho}_1\gamma_1$ and $\rho_2<\bar\rho_1,\bar\rho_2>\xi = \bar\rho_2\xi = \rho_2\hat{\bar\rho}_1\gamma_1$ it follows that $\hat\rho_1<<\bar\rho_1,\bar\rho_2>,a\bar\rho_3>\xi = <\bar\rho_1,\bar\rho_2>\xi = \hat\rho_1\gamma_1$.

Now since $\theta_1((a\times a)\times 1_X)\gamma_1 = \theta_2(1_X\times(a\times a))\gamma_2$ it follows that

$$\hat\rho_2\gamma_1 = \hat\pi_2{}^*((a\times a)\times 1_X)\gamma_1 = \bar\pi_3\theta_1((a\times a)\times 1_X)\gamma_1 = \bar\pi_3\theta_2(1_X\times(a\times a))\gamma_2 =$$
$$\pi_2\pi_2{}^*(1_X\times(a\times a))\gamma_2 = \pi_2(a\times a)\tilde\rho_2\gamma_2 = a\rho_2\tilde\rho_2\gamma_2.$$

Whence $\hat\rho_2\gamma_1 = \hat\rho_2<<\bar\rho_1,\bar\rho_2>,a\rho_3>\xi = a\rho_3\xi = a\rho_2\tilde\rho_2\gamma_2$. Thus

$$<<\bar\rho_1,\bar\rho_2>,a\rho_3>\xi = \gamma_1.$$

Again since $\theta_1((a\times a)\times 1_X)\gamma_1 = \theta_2(1_X\times(a\times a))\gamma_2$, it follows that

$$\tilde\rho_1\gamma_2 = \pi_1{}^*(1_X\times(a\times a))\gamma_2 = \bar\pi_1\theta_2(1_X\times(a\times a))\gamma_2 = \bar\pi_1\theta_1((a\times a)\times 1_X)\gamma_1 =$$
$$\pi_1\hat\pi_1{}^*((a\times a)\times 1_X)\gamma_1 = \pi_1(a\times a)\hat\rho_1\gamma_1 = a\rho_1\hat\rho_1\gamma_1.$$

Hence $\tilde\rho_1<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>>\xi = a\bar\rho_1\xi = a\rho_1\hat{\bar\rho}_1\gamma_1 = \tilde\rho_1\gamma_2$.

Since $\rho_1<\bar\rho_2,\bar\rho_3>\xi = \bar\rho_2\xi = \rho_2\tilde\rho_2\gamma_2$ and $\rho_2<\bar\rho_2,\bar\rho_3>\xi = \bar\rho_3\xi = \rho_2\tilde\rho_2\gamma_2$ it follows that $<\bar\rho_2,\bar\rho_3>\xi = \tilde\rho_2\gamma_2$. Hence $\tilde\rho_2<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>>\xi = <\bar\rho_2,\bar\rho_3>\xi = \tilde\rho_2\gamma_2$. Consequently $<a\bar\rho_1,<\bar\rho_2,\bar\rho_3>>\xi = \gamma_2$.

From this it follows that $(a\times a\times a)\xi = \delta$. Since $(a\times a\times a)$ is a monomorphism the morphism ξ is unique. Thus it has been shown that $(A\times A\times A, a\times a\times a)$ is the intersection of $(X\times(A\times A),\theta_2(1_X\times(a\times a)))$ and

$((A\times A)\times X, \Theta_1((a\times a)\times 1_X))$. It next will be shown that the following diagram commutes.

$$
\begin{tikzcd}
A\times A\times A \arrow[rr, "a\times a\times a"] \arrow[dr, "\tau\#"] \arrow[ddr, "<\bar{\rho}_1,\bar{\rho}_3>"'] & & X\times X\times X \arrow[rr, "<\bar{\pi}_1,\bar{\pi}_3>"] & & X\times X \\
& (A\times A)\circ(A\times A) \arrow[urrr, "j\#"] & & & \\
& A\times A \arrow[uurrr, "a\times a"'] & & &
\end{tikzcd}
$$

$(\tau\#, j\#)$ is the epi-extremal mono factorization of $<\bar{\pi}_1,\bar{\pi}_3>(a\times a\times a)$. Now $\pi_1(a\times a)<\bar{\rho}_1,\bar{\rho}_3> = a\rho_1<\bar{\rho}_1,\bar{\rho}_3> = a\bar{\rho}_1 = \pi_1<\bar{\pi}_1,\bar{\pi}_3>(a\times a\times a)$ and $\pi_2(a\times a)<\bar{\rho}_1,\bar{\rho}_3> = a\rho_2<\bar{\rho}_1,\bar{\rho}_3> = a\bar{\rho}_3 = \pi_2<\bar{\pi}_1,\bar{\pi}_3>(a\times a\times a)$. Thus the above diagram commutes.

Since $((A\times A)\circ(A\times A), j\#)$ is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>(a\times a\times a)$ factors then

$$((A\times A)\circ(A\times A), j\#) \leq (A\times A, a\times a)$$

whence transitivity is obtained.

3.4. <u>Canonical Embedding</u>. Let (R,j) be a relation on X. Let (τ_1, j_1), (τ_2, j_2), and (τ_3, j_3) be the epi-extremal mono factorizations of $\pi_1 j$, $\pi_2 j$, and $\pi_2 j^*$ respectively. Let RX, XR, and XR^{-1} denote the domains of j_1, j_2, and j_3 (codomains of τ_1, τ_2, and τ_3) respectively.

$$
\begin{tikzcd}
& RX \arrow[dr, "j_1"] & \\
R \arrow[ur, "\tau_1"] \arrow[r, "j"] \arrow[d, "\tau"'] & X\times X \arrow[r, "\pi_1"] \arrow[d, "<\pi_2,\pi_1>"] & X \arrow[d, equal] \\
R^{-1} \arrow[r, "j^*"] \arrow[dr, "\tau_3"'] & X\times X \arrow[r, "\pi_2"] & X \\
& XR^{-1} \arrow[ur, "j_3"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
R \arrow[r, "j"] \arrow[dr, "\tau_2"'] & X \times X \arrow[r, "\pi_2"] & X \\
& XR \arrow[ur, "j_2"'] &
\end{tikzcd}$$

In the categories Set, FGp, Grp, Ab, (RX, j_1) may be taken to be the set $\{x \varepsilon X$: there exists $y \varepsilon X$ such that $(x,y) \varepsilon R\}$ together with the inclusion map. Similarly, in these same categories, (XR, j_2) may be taken to be the set $\{y \varepsilon X$: there exists $x \varepsilon X$ such that $(x,y) \varepsilon R\}$ together with the inclusion map and (XR^{-1}, j_3) may be taken to be the set $\{x \varepsilon X$: there exists $y \varepsilon X$ such that $(y,x) \varepsilon R^{-1}\}$ together with the inclusion map.

In the categories $\underline{\text{Top}}_1$, and $\underline{\text{Top}}_2$ the extremal subobjects (RX, j_1), (XR, j_2), and (XR^{-1}, j_3) of X have precisely the same underlying sets as above endowed with the subspace topology induced by the topology of X. See Section 4 (4.1, 4.2, and 4.3) for a more detailed discussion.

It is easy to see that in the category Set, a symmetric, transitive relation on a set X is an equivalence relation on a subset of X. Recall the discussion in Section 2 (2.0) of the remarks of Lambek who obtains the similar result for homomorphic relations on rings with identity. This result we wish to generalize. In order to do this we must first be able to pick out the subobject of X on which the relation is an equivalence relation.

Referring to the above diagrams, since τ is an isomorphism (1.9) and since the epi-extremal mono factorization is unique (0.18) it is clear that $(RX, j_1) \equiv (XR^{-1}, j_3)$. That is, there exists an isomorphism k from RX to XR^{-1} such that $j_3 k = j_1$ (see 4.4 and 4.5).

Consider the product $(RX{\times}XR,\tilde{\rho}_1,\tilde{\rho}_2)$. Also, consider the morphism $(j_1{\times}j_2)\langle\tau_1,\tau_2\rangle$ from R to X×X. Since

$\pi_1(j_1{\times}j_2)\langle\tau_1,\tau_2\rangle = j_1\tilde{\rho}_1\langle\tau_1,\tau_2\rangle = j_1\tau_1 = \pi_1 j$ and

$\pi_2(j_1{\times}j_2)\langle\tau_1,\tau_2\rangle = j_2\tilde{\rho}_2\langle\tau_1,\tau_2\rangle = j_2\tau_2 = \pi_2 j$ it follows from the definition of product that $(j_1{\times}j_2)\langle\tau_1,\tau_2\rangle = j$. Note, $\langle\tau_1,\tau_2\rangle$ is an extremal monomorphism since j is an extremal monomorphism (0.16).

Now suppose that (R,j) is symmetric on X. Then it follows that there exists an isomorphism α so that $j\alpha = j^*$ (1.13). Thus $\pi_2 j\alpha = \pi_2 j^*$ and this together with the fact that α is an isomorphism and the uniqueness of the epi-extremal mono factorization implies that $(XR,j_2) \equiv (XR^{-1},j_3)$. That is, there exists an isomorphism β so that $j_2\beta = j_3$. Thus it is routine to see that the following diagram commutes.

$$
\begin{tikzcd}
R \arrow[rr, "\tau_1"] \arrow[dr, "j"] & & XR \arrow[dr, "j_2"] & \\
& X{\times}X \arrow[rr, "\pi_2"] & & X \arrow[dd, equal] \\
R^{-1} \arrow[uu, "\alpha"] \arrow[ur, "j^*"] \arrow[rr, "\tau_3"] \arrow[dr, "j^*"] & & XR^{-1} \arrow[uu, "\beta"'] \arrow[ur, "j_3"] \arrow[dr, "j_3"] & \\
& X{\times}X \arrow[rr, "\pi_2"] & & X \\
R \arrow[uu, "\tau"] \arrow[ur, "\langle\pi_2,\pi_1\rangle j"'] \arrow[rr, "\tau_1"'] & & RX \arrow[uu, "k"'] \arrow[ur, "j_1"'] &
\end{tikzcd}
$$

Consider the following products: $(XR\times XR,\rho_1,\rho_2)$ and $(X\times X,\pi_1,\pi_2)$. Letting $\psi = (\beta k\times 1_{XR})<\tau_1,\tau_2>$ then $(j_2\times j_2)\psi = j$, since $\pi_1(j_2\times j_2)\psi = j_2\rho_1\psi = j_2\beta k\tilde{\rho}_1<\tau_1,\tau_2> = j_2\beta k\tau_1 = j_1\tau_1 = \pi_1 j$ and $\pi_2(j_2\times j_2)\psi = j_2\rho_2\psi = j_2 1_{XR}\tilde{\rho}_2<\tau_1,\tau_2> = j_2\tau_2 = \pi_2 j$. Thus the following diagram commutes and the relation (R,ψ) on XR shall be called the <u>canonical embedding of</u> R <u>into</u> $XR\times XR$.

$$\begin{tikzcd}
R \arrow[rr, "<\tau_1,\tau_2>"] \arrow[drr, "\psi"] \arrow[ddrr, "j"'] & & RX\times XR \arrow[d, "\beta k\times 1_{XR}"] \\
& & XR\times XR \arrow[d, "j_2\times j_2"] \\
& & X\times X
\end{tikzcd}$$

3.5. <u>Lemma</u>. Let (R,j) be a symmetric relation on X. Then (R,ψ) is a symmetric relation on XR.

<u>Proof</u>. Suppose that (R,j) is symmetric on X and let $(\tilde{R},\psi^*)$ be the inverse of (R,ψ) on XR. Then $<\rho_2,\rho_1>\psi = \psi^*\tau^*$. It is easy to verify that $<\pi_2,\pi_1>j = (j_2\times j_2)<\rho_2,\rho_1>\psi$. Thus since (R^{-1},j^*) is the intersection of all extremal subobjects through which $<\pi_2,\pi_1>j$ factors there exists a morphism λ from R^{-1} to R so that $(j_2\times j_2)\psi^*\lambda = j^*$. But $j^*\tau = <\pi_2,\pi_1>j$ whence $(j_2\times j_2)\psi^*\tau^* = (j_2\times j_2)<\rho_2,\rho_1>\psi = <\pi_2,\pi_1>j = j^*\tau$. So $j^*\tau = ((j_2\times j_2)\psi^*)\lambda\tau = ((j_2\times j_2)\psi^*)\tau^*$. Since $(j_2\times j_2)\psi^*$ is a monomorphism it follows that $\lambda\tau = \tau^*$. Recall that τ and τ^* both are isomorphisms (1.9). Hence λ is an isomorphism.

Recall that by the definition of symmetry (1.10), there exists a morphism α from R^{-1} to R so that $j\alpha = j^*$. Thus $(j_2 \times j_2)\psi^*\lambda\alpha^{-1} = j^*\alpha^{-1} = j = (j_2\times j_2)\psi$. But since $(j_2\times j_2)$ is a monomorphism this implies that $\psi^*\lambda\alpha^{-1} = \psi$. Thus since λ and α are isomorphisms, we have $(\tilde{R},\psi^*) \equiv (R,\psi)$. Hence (R,ψ) is symmetric on XR.

3.6. <u>Lemma</u>. If (R,j) is a quasi-equivalence on X then (R,ψ) is a quasi-equivalence on XR.

<u>Proof</u>. In view of Lemma 3.5 it need only be shown that (R,ψ) is transitive on XR. To that end first consider the following diagram. It will be shown that there exists a morphism λ such that the diagram commutes.

$$\begin{tikzcd}
R\times XR \arrow[r, tail, "\psi\times 1_{XR}"] \arrow[ddd, bend right, "1_R\times j_2"'] & (XR\times XR)\times XR \arrow[dr, "\tilde{\theta}_1"] & & \\
(R\times XR)\cap(XR\times R) \arrow[u, "\lambda_1"'] \arrow[d, "\lambda_2"] \arrow[rr, "\delta"] \arrow[ddd, dashed, "\lambda"] & & XR\times XR\times XR \arrow[r, "\langle\bar{\rho}_1,\bar{\rho}_3\rangle"] \arrow[ddd, "j_2\times j_2\times j_2"] & XR\times XR \arrow[ddd, "j_2\times j_2"] \\
XR\times R \arrow[r, tail, "1_{XR}\times\psi"] \arrow[ddd, bend right, "j_2\times 1_R"'] & XR\times(XR\times XR) \arrow[ur, "\tilde{\theta}_2"'] & & \\
R\times X \arrow[r, tail, "j\times 1_X"] & (X\times X)\times X \arrow[dr, "\theta_1"] & & \\
(R\times X)\cap(X\times R) \arrow[u, "\lambda_3"'] \arrow[d, "\lambda_4"] \arrow[rr, tail, "\gamma"] & & X\times X\times X \arrow[r, "\langle\bar{\pi}_1,\bar{\pi}_3\rangle"] & X\times X \\
X\times R \arrow[r, tail, "1_X\times j"] & X\times(X\times X) \arrow[ur, "\theta_2"'] & &
\end{tikzcd}$$

Clearly $\langle\bar{\pi}_1,\bar{\pi}_3\rangle(j_2\times j_2\times j_2) = (j_2\times j_2)\langle\bar{\rho}_1,\bar{\rho}_3\rangle$. Also,

$$\Theta_1(j\times 1_X)(1_R\times j_2)\lambda_1 = \Theta_1(j\times j_2)\lambda_1 = \Theta_1((j_2\times j_2)\psi\times j_2)\lambda_1 =$$

$$\Theta_1((j_2\times j_2)\times j_2)(\psi\times 1_{XR})\lambda_1 = (j_2\times j_2\times j_2)\tilde{\Theta}_1(\psi\times 1_{XR})\lambda_1 \text{ and}$$

$$\Theta_2(1_X\times j)(j_2\times 1_R)\lambda_2 = \Theta_2(j_2\times j)\lambda_2 = \Theta_2(j_2\times(j_2\times j_2)\psi)\lambda_2 =$$

$$\Theta_2(j_2\times(j_2\times j_2))(1_{XR}\times\psi)\lambda_2 = (j_2\times j_2\times j_2)\tilde{\Theta}_2(1_{XR}\times\psi)\lambda_2$$

as can be verified in a straightforward manner. Thus the diagram above is commutative, and in particular,

$\Theta_1(j\times 1_X)(1_R\times j_2)\lambda_1 = (j_2\times j_2\times j_2)\delta = \Theta_2(1_X\times j)(j_2\times 1_R)\lambda_2$. Hence, by the definition of intersection there exists a unique morphism λ such that $\gamma\lambda = (j_2\times j_2\times j_2)\delta$.

Let $(\widetilde{RoR},\psi\#)$ denote the composition of (R,ψ) with (R,ψ) on XR. Let (RoR,j') be the composition of (R,j) with (R,j) on X. Then $\langle\bar{\rho}_1,\bar{\rho}_3\rangle\delta = \psi\#\tau\#$ where $\tau\#$ is an epimorphism and $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma = j'\tau'$ where τ' is an epimorphism. But since $\gamma\lambda = (j_2\times j_2\times j_2)\delta$, it follows that $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\lambda = (j_2\times j_2)\langle\bar{\rho}_1,\bar{\rho}_3\rangle\delta$ so that $(j_2\times j_2)\psi\#\tau\# = j'\tau'\lambda$. Hence the following diagram commutes.

$$\begin{tikzcd}
(R\times XR)\cap(XR\times R) \arrow[r, two heads, "\tau\#"] \arrow[d, "\lambda"] & \widetilde{RoR} \arrow[d, "\psi\#"] \arrow[ddl, dashed, "\xi"'] \\
(R\times X)\cap(X\times R) \arrow[d, two heads, "\tau'"] & XR\times XR \arrow[d, "j_2\times j_2"] \\
RoR \arrow[r, tail, "j'"'] & X\times X
\end{tikzcd}$$

Since $\tau\#$ is an epimorphism and j' is an extremal monomorphism, by the diagonalizing property (0.19) there exists a unique morphism ξ such that $j'\xi = (j_2\times j_2)\psi\#$ and $\xi\tau\# = \tau'\lambda$. But this says that $(\widetilde{R\circ R},(j_2\times j_2)\psi\#) \leq (R\circ R,j')$. Since (R,j) is transitive $(R\circ R,j') \leq (R,j)$ hence $(\widetilde{R\circ R},(j_2\times j_2)\psi\#) \leq (R,j) = (R,(j_2\times j_2)\psi)$. Hence there exists a morphism σ from $R\circ R$ to R such that $(j_2\times j_2)\psi\sigma = (j_2\times j_2)\psi\#$. Again, $j_2\times j_2$ is a monomorphism so that $\psi\sigma = \psi\#$ which says that $(\widetilde{R\circ R},\psi\#) \leq (R,\psi)$ hence (R,ψ) is transitive.

3.7. <u>Theorem</u>. If (R,j) is a relation from X to Y and $\pi_1 j$ is an epimorphism then $(R\circ R^{-1},j\#)$ is reflexive on X.

<u>Proof</u>. It will first be shown that the following diagram commutes.

$$\begin{tikzcd}
 & R\times X \arrow[r, tail, "j\times 1_X"] & (X\times Y)\times X \arrow[dr, tail, "\Theta_1"] & \\
R \arrow[ur, "<1_R,\pi_1 j>"] \arrow[r, dashed, "\Sigma"] \arrow[dr, "<\pi_1 j,\tau>"'] & (R\times X)\cap(X\times R^{-1}) \arrow[u, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[rr, tail, "\gamma"] & & X\times Y\times X \\
 & X\times R^{-1} \arrow[r, tail, "1_X\times j^*"'] & X\times(Y\times X) \arrow[ur, tail, "\Theta_2"'] &
\end{tikzcd}$$

Consider the following products: $(X\times Y,\pi_1,\pi_2)$, $(X\times Y\times X,\bar\pi_1,\bar\pi_2,\bar\pi_3)$, $(R\times X,\rho_1,\rho_2)$, $(X\times R^{-1},\rho_1{}^*,\rho_2{}^*)$, $((X\times Y)\times X,\pi_1{}^*,\pi_2{}^*)$, $(X\times(Y\times X),\tilde\pi_1,\tilde\pi_2)$, $(Y\times X,\hat\pi_1,\hat\pi_2)$, and $(X\times X,\bar{\bar\pi}_1,\bar{\bar\pi}_2)$.

Now,

$\bar\pi_1\Theta_1(j\times 1_X)<1_R,\pi_1 j> = \pi_1\pi_1{}^*(j\times 1_X)<1_R,\pi_1 j> = \pi_1 j\rho_1<1_R,\pi_1 j> = \pi_1 j.$

$\bar\pi_2\Theta_1(j\times 1_X)<1_R,\pi_1 j> = \pi_2\pi_1{}^*(j\times 1_X)<1_R,\pi_1 j> = \pi_2 j.$

$\bar\pi_3\Theta_1(j\times 1_X)<1_R,\pi_1 j> = \pi_2{}^*(j\times 1_X)<1_R,\pi_1 j> = \rho_2<1_R,\pi_1 j> = \pi_1 j.$

$\bar{\pi}_1\Theta_2(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \tilde{\pi}_1(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \rho_1^*\langle\pi_1 j,\tau\rangle = \pi_1 j.$

$\bar{\pi}_2\Theta_2(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \hat{\pi}_1\tilde{\pi}_2(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \hat{\pi}_1 j^*\rho^*\langle\pi_1 j,\tau\rangle =$

$$\hat{\pi}_1 j^*\tau = \hat{\pi}_1\langle\pi_2,\pi_1\rangle j = \pi_2 j.$$

$\bar{\pi}_3\Theta_2(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \hat{\pi}_2\tilde{\pi}_2(1_X\times j^*)\langle\pi_1 j,\tau\rangle = \hat{\pi}_2 j^*\tau = \hat{\pi}_2\langle\pi_2,\pi_1\rangle j = \pi_1 j.$

Thus by the definition of product the diagram commutes. Hence there exists a morphism Σ so that $\lambda_1\Sigma = \langle 1_R,\pi_1 j\rangle$ and $\lambda_2\Sigma = \langle\pi_1 j,\tau\rangle$.

From the above it is easy to see that

$\bar{\bar{\pi}}_1\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma = \bar{\pi}_1\gamma\Sigma = \pi_1 j = \bar{\pi}_3\gamma\Sigma = \bar{\bar{\pi}}_2\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma$. Recall that (Δ_X,i_X) is the equalizer of $\bar{\bar{\pi}}_1$ and $\bar{\bar{\pi}}_2$ hence there exists a morphism ϕ such that $i_X\phi = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma$.

Let $(R\circ R^{-1},j\#)$ be the indicated composition of relations and let $\tau\#$ denote that epimorphism for which $j\#\tau\# = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma$. Thus, combining the above results, $\langle\pi_1 j,\pi_1 j\rangle = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma = j\#\tau\#\Sigma = i_X\phi$.

Since (Δ_X,i_X) and $(X,\langle 1_X,1_X\rangle)$ are isomorphic as extremal subobjects of $X\times X$ (1.21), there exists an isomorphism λ such that $\langle 1_X,1_X\rangle\lambda = i_X$. Consequently, $\langle 1_X,1_X\rangle\lambda\phi = i_X\phi = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma = \langle\pi_1 j,\pi_1 j\rangle$.

Now $\bar{\bar{\pi}}_1\langle 1_X,1_X\rangle\lambda\phi = 1_X\lambda\phi = \lambda\phi = \bar{\bar{\pi}}_1\langle\pi_1 j,\pi_1 j\rangle = \pi_1 j$ and by hypothesis $\pi_1 j$ is an epimorphism; thus, since λ is an isomorphism, it follows that ϕ must be an epimorphism.

Thus $\langle\pi_1 j,\pi_1 j\rangle$ has (ϕ,i_X) as its epi-extremal mono factorization. But this means that (Δ_X,i_X) is the intersection of all extremal subobjects of $X\times X$ through which $\langle\pi_1 j,\pi_1 j\rangle$ factors (0.21). Recall that $\langle\pi_1 j,\pi_1 j\rangle = j\#\tau\#\Sigma$, thus $(\Delta_X,i_X) \leq (R\circ R^{-1},j\#)$ which was to be proved.

3.8. Corollary. If (R,j) is symmetric on X then $(\widetilde{R\circ R},\psi\#)$, the composition of (R,ψ) with (R,ψ) on XR, is reflexive on XR.

Proof. Since (R,j) is symmetric on X then (R,ψ) is symmetric on XR (3.5) hence $(R,\psi) \equiv (R^{-1},\psi^*)$ (1.13). Referring to the diagram in (3.4) fol-

lowing the definition of the canonical embedding it is immediate that $\rho_1\psi$ is an epimorphism since $\rho_1\psi = \beta k\tau_1$ and each of β, k, and τ_1 is an epimorphism. Thus $(R\circ R^{-1}, \tilde{\psi}) \equiv (\widetilde{R\circ R}, \psi\#)$ is reflexive on XR (3.7).

3.9. Corollary. If (R,j) is a quasi-equivalence on X then (R,j) is an equivalence relation if and only if $\pi_1 j$ is an epimorphism (respectively if and only if $\pi_2 j$ is an epimorphism).

Proof. If (R,j) is an equivalence relation then (R,j) is reflexive and a quasi-equivalence. Thus by Proposition 1.24, $\pi_1 j$ and $\pi_2 j$ are retractions hence epimorphisms.

Conversely, if $\pi_1 j$ is an epimorphism then applying the theorem and Proposition 1.30, $(\Delta_X, i_X) \leq (R\circ R^{-1}, j\#) \leq (R\circ R, j') \leq (R,j)$ so that (R,j) is reflexive and hence is an equivalence relation. (If $\pi_2 j$ is an epimorphism then $(\Delta_X, i_X) \leq (R^{-1}\circ R, j\#*) \leq (R^{-1}\circ R^{-1}, j'*) \leq (R^{-1}, j*)$ and $(R^{-1}, j*) \equiv (R,j)$.)

3.10. Corollary. If (R,j) is a quasi-equivalence on X then (R,ψ) is an equivalence relation on XR.

Proof. (R,ψ) is a quasi-equivalence on XR (3.6) and $(\widetilde{R\circ R}, \psi\#)$ is reflexive on XR (3.8). Thus $(\Delta_{XR}, i_{XR}) \leq (\widetilde{R\circ R}, \psi\#) \leq (R,\psi)$ whence (R,ψ) is reflexive and thus is an equivalence relation on XR.

3.11. Proposition. If (R,j) is a quasi-equivalence on X then (R,j) and $(R\circ R, j')$ are isomorphic relations on X.

Proof. By Corollary 3.10 (R,ψ) is an equivalence relation on XR whence $(\widetilde{R\circ R}, \psi\#)$ and (R,ψ) are isomorphic relations on XR (1.32). Recall that there exists a morphism ξ such that the following diagram commutes (3.6).

$$\begin{tikzcd}
(R\times XR)\cap(XR\times R) \arrow[r, "\tau\#"] \arrow[d, "\tau'\lambda"'] & \widetilde{RoR} \arrow[d, "(j_2\times j_2)\psi\#"] \arrow[dl, dashed, "\xi"'] \\
RoR \arrow[r, "j'"'] & X\times X
\end{tikzcd}$$

Thus $(\widetilde{RoR},(j_2\times j_2)\psi\#) \leq (RoR,j')$. But as mentioned above $(\widetilde{RoR},\psi\#) \equiv (R,\psi)$ hence there exists an isomorphism $\lambda\#$ such that $\psi\#\lambda\# = \psi$. So by the definition of the canonical embedding (3.4), $j'\xi\lambda\# = (j_2\times j_2)\psi\#\lambda\# = (j_2\times j_2)\psi = j$. But this implies that $(R,j) \leq (RoR,j')$. Thus since (R,j) is transitive, $(R,j) \doteq (RoR,j')$ which was to be proved.

3.12. Proposition. Let (R,j) be a relation on X. Then $(R,j) \leq (\Delta_X,i_X)$ if and only if R is symmetric on X and $(R,\psi) \leq (\Delta_{XR},i_{XR})$.

Proof. If $(R,j) \leq (\Delta_X,i_X)$ then there exists a morphism α such that $j = i_X\alpha$. Thus $\pi_1 j = \pi_1 i_X\alpha = \pi_2 i_X\alpha = \pi_2 j$ whence $\pi_1<\pi_2,\pi_1>j = \pi_2 j = \pi_1 j = \pi_2<\pi_2,\pi_1>j$. Thus by the definition of product $<\pi_2,\pi_1>j = j$. Consequently the epi-extremal mono factorization of $<\pi_2,\pi_1>j$ is $(1_R,j)$ and so $(R,j) \equiv (R^{-1},j^*)$; i.e., (R,j) is symmetric.

Recall that $j = (j_2\times j_2)\psi$ (3.4). Thus $\pi_1 j = \pi_1(j_2\times j_2)\psi = j_2\rho_1\psi$ and $\pi_2 j = \pi_2(j_2\times j_2)\psi = j_2\rho_2\psi$. But $\pi_1 j = \pi_2 j$ hence $j_2\rho_1\psi = j_2\rho_2\psi$. Since j_2 is a monomorphism it follows that $\rho_1\psi = \rho_2\psi$. Recall that (Δ_{XR},i_{XR}) is the equalizer of ρ_1 and ρ_2. Hence there exists a morphism β such that $i_{XR}\beta = \psi$. This implies that $(R,\psi) \leq (\Delta_{XR},i_{XR})$.

Conversely, if R is symmetric and $(R,\psi) \leq (\Delta_{XR},i_{XR})$ then there exists a morphism $\tilde{\beta}$ such that $\psi = i_{XR}\tilde{\beta}$; hence

$\rho_1\psi = \rho_1 i_{XR}\tilde{\beta} = \rho_2 i_{XR}\tilde{\beta} = \rho_2\psi$. Since $(j_2 \times j_2)\psi = j$, we have $\pi_1 j = \pi_1(j_2 \times j_2)\psi = j_2\rho_1\psi = j_2\rho_2\psi = \pi_2(j_2 \times j_2)\psi = \pi_2 j$. Thus $\pi_1 j = \pi_2 j$ so that there exists a morphism $\tilde{\alpha}$ such that $j = i_X\tilde{\alpha}$. This means that $(R,j) \leq (\Delta_X, i_X)$.

3.13. Definition. Let (R,j) be a relation on X. Then R is said to be a circular relation if and only if $R \circ R \leq R^{-1}$.

This notion is due to MacLane and Birkhoff [20] (exercize 3, page 14).

3.14. Proposition. Let (R,j) be a relation on X. Then R is a circular relation if and only if R^{-1} is a circular relation.

Proof. If R is circular then $R \circ R \leq R^{-1}$. Thus $R^{-1} \circ R^{-1} \equiv (R \circ R)^{-1} \leq (R^{-1})^{-1} \equiv R$ (1.38, 1.12 and 1.11). Hence R^{-1} is circular.

Conversely, if R^{-1} is circular then by the above, $(R^{-1})^{-1} \equiv R$ is circular.

3.15. Theorem. Let (R,j) be a relation on X. Then R is an equivalence relation on X if and only if R is reflexive and circular.

Proof. If R is an equivalence relation then R is reflexive. Since R is transitive and symmetric, $R \circ R \leq R \equiv R^{-1}$ hence R is circular.

Conversely, if R is reflexive and circular then R^{-1} is reflexive (1.17) and R^{-1} is circular (3.14). Hence $R^{-1} \equiv R^{-1} \circ \Delta_X \leq R^{-1} \circ R^{-1} \leq R$ (1.31 and 1.12) whence R is symmetric. Thus $R \equiv R^{-1}$ (1.11).

Now $R \circ R \equiv R^{-1} \circ R^{-1} \leq R$ hence R is transitive. Thus R is an equivalence relation.

SECTION 4. IMAGES

4.1. Definition. Let (R,j) be a relation from X to Y and let (A,a) and (B,b) be extremal subobjects of X and Y respectively. Consider $(R\cap(A\times Y),\gamma)$ and $(R\cap(X\times B),\delta)$. Let $(\tilde{\tau}_1,\alpha)$ and $(\tilde{\tau}_2,\beta)$ be the epi-extremal mono factorizations of $\pi_2\gamma$ and $\pi_1\delta$ respectively. Denote the domain of α by AR and the domain of β by RB. Thus the following diagrams commute.

$$\begin{tikzcd}
 & R \arrow[dr, tail, "j"] & & & \\
R\cap(A\times Y) \arrow[ur, "\lambda_1"] \arrow[rr, tail, "\gamma"] \arrow[dr, "\lambda_2"'] \arrow[drrr, two heads, "\tilde{\tau}_1" near start] & & X\times Y \arrow[rr, "\pi_2"] & & Y \\
 & A\times Y \arrow[ur, tail, "a\times 1_Y"'] & & AR \arrow[ur, tail, "\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & R \arrow[dr, tail, "j"] & & & \\
R\cap(X\times B) \arrow[ur, "\lambda_3"] \arrow[rr, tail, "\delta"] \arrow[dr, "\lambda_4"'] \arrow[drrr, two heads, "\tilde{\tau}_2" near start] & & X\times Y \arrow[rr, "\pi_1"] & & X \\
 & X\times B \arrow[ur, tail, "1_X\times b"'] & & RB \arrow[ur, tail, "\beta"'] &
\end{tikzcd}$$

4.2. Remark. Since $(X,1_X)$ and $(Y,1_Y)$ are extremal subobjects of X and Y respectively, then $(R\cap(X\times Y),\gamma)\equiv(R,j)$ and $(R\cap(X\times Y),\delta)\equiv(R,j)$ whence (XR,α) is precisely the extremal subobject (XR,j_2) used in the canonical embedding (3.4). Since $X = Y$ in 3.4 then also (RY,β) is precisely

(RX, j_1) used in 3.4.

4.3. Example. In the category Set, for $(A,a) \leq (X, 1_X)$, $(B,b) \leq (Y, 1_Y)$ and $(R,j) \leq (X\times Y, 1_{X\times Y})$,

$$AR \cong \{y\varepsilon Y: \text{there exists } a\varepsilon A \text{ such that } (a,y)\varepsilon R\}$$

$$RB \cong \{x\varepsilon X: \text{there exists } b\varepsilon B \text{ such that } (x,b)\varepsilon R\}.$$

This is easily seen since $R\cap(A\times Y) \cong \{(a,y): a\varepsilon A,\ (a,y)\varepsilon R\}$ and $R\cap(X\times B) \cong \{(x,b): b\varepsilon B,\ (x,b)\varepsilon R\}$, and AR is the set of all second terms of elements of $R\cap(A\times Y)$ and RB is the set of all first terms of elements of $R\cap(X\times B)$.

In the category $\underline{\text{Top}}_1$, AR and RB have precisely the same underlying sets as above. They are endowed with the subspace topology determined by the topology of $X\times Y$.

In the category $\underline{\text{Top}}_2$, AR and RB have precisely the same underlying sets as in $\underline{\text{Top}}_1$ for it is easy to verify that AR and RB are closed subsets of X and Y respectively. Recall that the image of a morphism in $\underline{\text{Top}}_2$ is the closure of the set theoretic image (0.15).

4.4. Theorem. If (R,j) is a relation from X to Y and (A,a) is an extremal subobject of X then (AR,α) and $(R^{-1}A,\beta)$ are isomorphic extremal subobjects of Y.

Proof. Consider the following commutative diagrams.

$$\begin{tikzcd}
& R \arrow[dr, tail, "j"] & & & \\
R\cap(A\times Y) \arrow[ur, "\lambda_1"] \arrow[rr, tail, "\gamma"] \arrow[dr, "\lambda_2"'] \arrow[drrr, two heads, "\tilde{\tau}_1" near start] & & X\times Y \arrow[rr, "\pi_2"] & & Y \\
& A\times Y \arrow[ur, tail, "a\times 1_Y"'] & & AR \arrow[ur, tail, "\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & R^{-1} \arrow[drr, "j^*"] & & & \\
R^{-1}\cap(Y\times A) \arrow[ur, "\lambda_3"] \arrow[rr, "\delta"] \arrow[rrrd, two heads, "\tilde{\tau}_2"] \arrow[dr, "\lambda_4"'] & & Y\times X \arrow[rr, "\hat{\pi}_1"] & & Y \\
 & Y\times A \arrow[ur, tail, "1_Y\times a"'] & & R^{-1}A \arrow[ur, tail, "\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[rr, tail, "j"] \arrow[dr, two heads, "\tau"'] & & X\times Y \arrow[rr, "<\pi_2,\pi_1>"] & & Y\times X \\
 & & R^{-1} \arrow[urr, tail, "j^*"'] & &
\end{tikzcd}$$

It can be shown in a straightforward manner that

$$<\pi_2,\pi_1>(a\times 1_Y) = (1_Y\times a)<\rho_2,\rho_1>$$

where ρ_1 and ρ_2 are the projections of $A\times Y$. Hence $(1_Y\times a)<\rho_2,\rho_1>\lambda_2 = <\pi_2,\pi_1>(a\times 1_Y)\lambda_2 = <\pi_2,\pi_1>\gamma = <\pi_2,\pi_1>j\lambda_1 = j^*\tau\lambda_1$. Thus by the definition of intersection there exists a morphism ξ such that $\delta\xi = <\pi_2,\pi_1>\gamma = j^*\tau\lambda_1 = (1_Y\times a)<\rho_2,\rho_1>\lambda_2$. Hence $\hat{\pi}_1\delta\xi = \hat{\pi}_1<\pi_2,\pi_1>\gamma = \pi_2\gamma = \alpha\tilde{\tau}_1$. But $\hat{\pi}_1\delta\xi = \beta\tilde{\tau}_2\xi$. Thus, since (AR,α) is the intersection of all extremal subobjects through which $\pi_2\gamma$ factors (0.21), it follows that $(AR,\alpha) \leq (R^{-1}A,\beta)$.

Similarly, it follows that $<\pi_2,\pi_1>^{-1}j^*\lambda = <\pi_2,\pi_1>^{-1}\delta = j\tau^{-1}\lambda_3 = (a\times 1_Y)<\rho_2,\rho_1>^{-1}\lambda_4$ whence there exists a morphism ξ^* such that $\gamma\xi^* = <\pi_2,\pi_1>^{-1}\delta$. Then $\pi_2<\pi_2,\pi_1>^{-1}\delta = \hat{\pi}_1\delta = \beta\tau_2 = \pi_2\gamma\xi^* = \alpha\tilde{\tau}_1\xi^*$. Again, since $(R^{-1}A,\beta)$ is the intersection of all extremal subobjects through which $\hat{\pi}_1\delta$ factors,

$(R^{-1}A,\beta) \leq (AR,\alpha)$. Consequently $(R^{-1}A,\beta) \equiv (AR,\alpha)$.

4.5. Corollary. If (R,j) is a relation from X to Y and (B,b) is an extremal subobject of Y then (RB,β) and (BR^{-1},α) are isomorphic as extremal subobjects of X.

Proof. Recall $((R^{-1})^{-1},j\#) \equiv (R,j)$ (1.11). Letting (R^{-1},j^*) play the role of (R,j) and (B,b) the role of (A,a) in the theorem, the following is obtained: $(BR^{-1},\alpha) \equiv ((R^{-1})^{-1}B,\beta\#) \equiv (RB,\beta)$.

4.6. Corollary. If (R,j) is a symmetric relation on X and (A,a) is an extremal subobject of X then (AR,α) and (RA,β) are isomorphic as extremal subobjects of X. (In particular, (XR,j_2) and (RX,j_1) are isomorphic as extremal subobjects of X as was shown directly in 3.4.)

Proof. Recall that $(R^{-1},j^*) \equiv (R,j)$ (1.13). Hence by the theorem

$$(AR,\alpha) \equiv (R^{-1}A,\tilde{\beta}) \equiv (RA,\beta).$$

4.7. Proposition. Let (A_1,a_1) and (A_2,a_2) be extremal subobjects of X and (R,j) be a relation from X to Y. If $(A_1,a_1) \leq (A_2,a_2)$ then

$$(A_1R,\alpha_1) \leq (A_2R,\alpha_2).$$

Proof. By hypothesis there exists a morphism μ so that $a_2\mu = a_1$. Thus, there exists a morphism ξ such that the following diagram commutes.

$$\begin{tikzcd}
 & R \arrow[dr, tail, "j"] & \\
R\cap(A_1\times Y) \arrow[ur] \arrow[rr, "\gamma_1"] \arrow[r] \arrow[d, "\xi"'] & A_1\times Y \arrow[r, "a_1\times 1_Y"'] \arrow[dd, "\mu\times 1_Y"] & X\times Y \arrow[d, equal] \\
R\cap(A_2\times Y) \arrow[rr, "\gamma_2"] \arrow[dr] \arrow[ddr] & & X\times Y \\
 & A_2\times Y \arrow[ur, "a_2\times 1_Y"] & \\
 & R \arrow[uur, tail, "j"'] &
\end{tikzcd}$$

Thus $\pi_2\gamma_1 = \pi_2\gamma_2\xi$ whence, because (A_1R,α_1) is the intersection of all extremal subobjects through which $\pi_2\gamma_1$ factors and $\pi_2\gamma_2\xi$ factors through (A_2R,α_2), $(A_1R,\alpha_1) \leq (A_2R,\alpha_2)$ which was to be proved.

4.8. Proposition. Let (B_1,b_1) and (B_2,b_2) be extremal subobjects of Y and (R,j) be a relation from X to Y. If $(B_1,b_1) \leq (B_2,b_2)$ then

$$(RB_1,\beta_1) \leq (RB_2,\beta_2).$$

Proof. $(RB_1,\beta_1) \equiv (B_1R^{-1},\beta_1{}^*) \leq (B_2R^{-1},\beta_2{}^*) \equiv (RB_2,\beta_2)$ (4.5 and 4.7).

4.9. Proposition. Let (R,j) and (S,k) be relations from X to Y and (A,a) be an extremal subobject of X. If $(R,j) \leq (S,k)$ then $(AR,\alpha) \leq (AS,\tilde{\alpha})$.

Proof. In a manner similar to that in the proof of 4.7 one can establish the existance of a morphism ξ such that the following diagram commutes.

$$\begin{tikzcd}
 & R \arrow[rr, "\mu"] \arrow[drrr, "j"] & & S \arrow[dr, "k"] & \\
R\cap(A\times Y) \arrow[ur] \arrow[rr, "\xi"] \arrow[drrr] & & S\cap(A\times Y) \arrow[ur] \arrow[rr, "\tilde{\gamma}"] \arrow[dr] & & X\times Y \\
 & & & A\times Y \arrow[ur, "a\times 1_Y"'] &
\end{tikzcd}$$

Hence the following diagram commutes.

$$\begin{tikzcd}
R\cap(A\times Y) \arrow[d, "\xi"'] \arrow[dr, "\gamma"] & \\
S\cap(A\times Y) \arrow[r, "\tilde{\gamma}"'] & X\times Y
\end{tikzcd}$$

Thus, since (AR,α) is the intersection of all extremal subobjects through which $\pi_2\gamma$ factors, and $\pi_2\tilde{\gamma}$ factors through $(AS,\tilde{\alpha})$, it follows that $(AR,\alpha) \leq (AS,\tilde{\alpha})$ which was to be proved.

4.10. <u>Proposition</u>. Let (R,j) and (S,k) be relations from X to Y and (B,b) be an extremal subobject of Y. If $(R,j) \leq (S,k)$ then

$$(RB,\beta) \leq (SB,\tilde{\beta}).$$

<u>Proof</u>. $(RB,\beta) \equiv (BR^{-1},\beta^*) \leq (BS^{-1},\tilde{\beta}^*) \equiv (SB,\tilde{\beta})$ (4.5, 1.12, and 4.9).

4.11. <u>Proposition</u>. Let (R,j) be a relation from X to Y and let (A_1,a_1) and (A_2,a_2) be extremal subobjects of X. Then

$$((A_1 \cap A_2)R,\alpha) \leq (A_1R \cap A_2R,\tilde{\alpha}).$$

<u>Proof</u>. Since $(A_1 \cap A_2,a) \leq (A_1,a_1)$ and $(A_1 \cap A_2,a) \leq (A_2,a_2)$ it follows that $((A_1 \cap A_2)R,\alpha) \leq (A_1R,\alpha_1)$ and $((A_1 \cap A_2)R,\alpha) \leq (A_2R,\alpha_2)$ (4.7). Thus

$$((A_1 \cap A_2)R,\alpha) \leq (A_1R \cap A_2R,\tilde{\alpha}) \text{ (1.19)}.$$

4.12. <u>Proposition</u>. Let (R,j) be a relation from X to Y and let (B_1,b_1) and (B_2,b_2) be extremal subobjects of Y. Then

$$(R(B_1 \cap B_2),\beta) \leq (RB_1 \cap RB_2,\tilde{\beta}).$$

<u>Proof</u>. $(R(B_1 \cap B_2),\beta) \equiv ((B_1 \cap B_2)R^{-1},\beta^*) \leq (B_1R^{-1} \cap B_2R^{-1},\tilde{\beta}^*) \equiv$

$$(RB_1 \cap RB_2,\tilde{\beta}) \text{ (4.5 and 4.11)}.$$

4.13. <u>Proposition</u>. Let (R_1,j_1) and (R_2,j_2) be relations from X to Y and let (A,a) be an extremal subobject of X. Then

$$(A(R_1 \cap R_2),\alpha) \leq (AR_1 \cap AR_2,\tilde{\alpha}).$$

<u>Proof</u>. It is clear that there exist morphisms ξ_1 and ξ_2 such that the following diagram commutes.

$$\begin{tikzcd}
& R_1 \cap (A\times Y) \arrow[dr, tail, "\gamma_1"] & \\
(R_1 \cap R_2) \cap (A\times Y) \arrow[ur, "\xi_1"] \arrow[rr, tail, "\gamma"] \arrow[dr, "\xi_2"'] & & X\times Y \\
& R_2 \cap (A\times Y) \arrow[ur, tail, "\gamma_2"'] &
\end{tikzcd}$$

Thus $\pi_2\gamma = \pi_2\gamma_1\xi_1 = \pi_2\gamma_2\xi_2$. Again since $(A(R_1 \cap R_2),\alpha)$ is the intersection of all extremal subobjects through which $\pi_2\gamma$ factors it follows that $(A(R_1\cap R_2),\alpha) \leq (AR_1,\alpha_1)$ and $(A(R_1\cap R_2),\alpha) \leq (AR_2,\alpha_2)$. Hence

$$(A(R_1\cap R_2),\alpha) \leq (AR_1 \cap AR_2,\tilde{\alpha}) \quad (1.19).$$

4.14. Proposition. Let (R_1,j_1) and (R_2,j_2) be relations from X to Y and let (B,b) be an extremal subobject of Y. Then

$$((R_1\cap R_2)B,\beta) \leq (R_1B \cap R_2B,\tilde{\beta}).$$

Proof. $((R_1\cap R_2)B,\beta) \equiv (B(R_1\cap R_2)^{-1},\beta^*) \leq (BR_1^{-1} \cap BR_2^{-1},\tilde{\beta}^*) \equiv$

$$(R_1B\cap R_2B,\tilde{\beta}) \quad (4.5 \text{ and } 4.13).$$

4.15. Proposition. Let (R,j) be a relation from X to Y then (R,j) and $(R\cap(RY\times Y),\gamma)$ are isomorphic as extremal subobjects of $X\times Y$.

Proof. Consider the following commutative diagrams.

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[d, two heads, "\tau_1"'] & X\times Y \arrow[d, "\pi_1"] \\
RY \arrow[r, tail, "j_1"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & RY\times Y \arrow[dr, tail, "j_1\times 1_Y"] & \\
R\cap(RY\times Y) \arrow[ur] \arrow[rr, tail, "\gamma"] \arrow[dr] & & X\times Y \\
 & R \arrow[ur, tail, "j"'] &
\end{tikzcd}$$

Since $(j_1\times 1_Y)\langle\tau_1,\pi_2 j\rangle = \langle\pi_1 j,\pi_2 j\rangle = j$, there exists a morphism ξ such that $\gamma\xi = j1_R$.

$$\begin{tikzcd}
 & & RY\times Y \arrow[drr, "j_1\times 1_Y"] & & \\
R \arrow[urr, "\langle \pi_1,\pi_2 j\rangle"] \arrow[r, "\xi"] \arrow[drr, "1_R"'] & R\cap(RY\times Y) \arrow[ur] \arrow[dr] \arrow[rrr, "\gamma"] & & & X\times Y \\
 & & R \arrow[urr, "j"'] & &
\end{tikzcd}$$

Thus $(R,j) \leq (R\cap(RY\times Y),\gamma)$. Clearly the reverse inequality holds so that $(R,j) \equiv (R\cap(RY\times Y),\gamma)$.

4.16. <u>Proposition</u>. Let (R,j) be a relation from X to Y. Then (R,j) and $(R\cap(X\times XR),\delta)$ are isomorphic relations from X to Y.

<u>Proof</u>. Analogous to the proof of 4.15.

4.17. <u>Corollary</u>. Let (R,j) be a relation from X to Y. Then (R,j) and $(R\cap(RY\times XR),\beta)$ are isomorphic relations from X to Y.

<u>Proof</u>. $(R,j) \equiv (R\cap(RY\times Y),\gamma) \equiv (R\cap(X\times XR),\delta)$ (4.15 and 4.16). But since (RY,j_1) and (XR,j_2) are extremal subobjects of X and Y respectively it follows that $((RY\times Y)\cap(X\times XR),\alpha) \equiv (RY\times XR,\beta)$. Thus $(R,j) \equiv ((R\cap(RY\times Y))\cap(R\cap(X\times XR)),\tilde{\beta}) \equiv (R\cap(RY\times XR),\beta)$.

4.18. <u>Proposition</u>. Let (R,j) be a relation from X to Y and let (A,a) be an extremal subobject of X. Then $(AR,\alpha) \equiv ((RY\cap A)R,\tilde{\alpha})$.

<u>Proof</u>. It follows from Proposition 1.5 that $R\cap((RY\cap A)\times Y)$ and $R\cap((RY\times Y)\cap(A\times Y))$ are isomorphic relations from X to Y. By Proposition 4.15, (R,j) and $(R\cap(RY\times Y,\gamma)$ are isomorphic relations from X to Y. Thus $R\cap(A\times Y)$ and $R\cap((RY\cap A)\times Y)$ are isomorphic relations from X to Y. Consequently by the definition of image (4.1), (AR,α) and $((RY\cap A)R,\tilde{\alpha})$ are isomorphic as extremal subobjects of Y.

4.19. <u>Corollary</u>. Let (R,j) be a relation from X to Y. Then $((RY)R,\alpha)$ and (XR,j_2) are isomorphic as extremal subobjects of Y.

Proof. Let $(X,1_X)$ play the role of (A,a) in 4.18.

4.20. Corollary. Let (R,j) be a relation from X to Y and let (B,b) be an extremal subobject of Y. Then BR^{-1} and $(B\cap XR)R^{-1}$ are isomorphic as extremal subobjects of X.

Proof. Immediate.

4.21. Proposition. Let (R,j) be a relation from X to Y. Then $(RoR^{-1},j\#)$ and $(RoR^{-1}\cap(RY\times X),\tilde{\gamma})$ are isomorphic relations on X.

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & & & RY\times X \arrow[dr, "j_1\times 1"] & \\
 & R\times X \arrow[urr, "\tau_1\times 1"] \arrow[r, tail, "j\times 1"'] & (X\times Y)\times X \arrow[d, "\theta_1"] & & \\
(R\times X)\cap(X\times R^{-1}) \arrow[ur, "\lambda_1"] \arrow[rr, tail, "\gamma"] \arrow[dr, "\lambda_2"] \arrow[ddr, bend right, "\tau\#"'] & & X\times Y\times X \arrow[rr, "<\pi_1,\pi_3>"] & & X\times X \\
 & X\times R^{-1} \arrow[r, tail, "1\times j^*"'] & X\times(Y\times X) \arrow[u, "\theta_2"'] & & \\
 & RoR^{-1} \arrow[uurrr, "j\#"'] & & &
\end{tikzcd}$$

To see the diagram is commutative it need only be observed that $(j_1\times 1)(\tau_1\times 1) = <\bar{\pi}_1,\bar{\pi}_3>\theta_1(j\times 1)$. To show this note that

$(j_1\times 1)(\tau_1\times 1) = (j_1\tau_1\times 1) = (\pi_1 j\times 1)$ and

$\pi_1<\bar{\pi}_1,\bar{\pi}_3>\theta_1(j\times 1) = \bar{\pi}_1\theta_1(j\times 1) = \pi_1 j\rho_1 = \pi_1(\pi_1 j\times 1)$,

$\pi_2<\bar{\pi}_1,\bar{\pi}_3>\theta_1(j\times 1) = \bar{\pi}_3\theta_1(j\times 1) = \rho_2 = \pi_2(\pi_1 j\times 1)$.

Thus, since $(RoR^{-1},j\#)$ is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>\gamma$ factors, it follows that

$(RoR^{-1}, j\#) \leq (RY\times X, j_1\times 1)$. Whence $(RoR^{-1}, j\#) \leq (RoR^{-1} \cap (RY\times X), \tilde{\gamma})$.

4.22. Theorem. Let (R,j) be a relation from X to Y. Then (RY, j_1) and $((RoR^{-1})X, \beta)$ are isomorphic as extremal subobjects of X.

Proof. Consider the following products: $(X\times Y\times X, \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$, $(X\times(Y\times X), \tilde{\rho}_1, \tilde{\rho}_2)$, $((X\times Y)\times X, \rho_1{}^*, \rho_2{}^*)$, $(R\times X, \rho_1, \rho_2)$, and $(X\times R^{-1}, \bar{\bar{\pi}}_1, \bar{\bar{\pi}}_2)$.

Referring to the diagram in the proof of 4.21 it is easy to see that: $\pi_1 <\bar{\pi}_1, \bar{\pi}_3> \gamma = \bar{\pi}_1 \theta_1 (j\times 1)\lambda_1 = \pi_1 \rho_1{}^*(j\times 1)\lambda_1 = \pi_1 j \rho_1 \lambda_1$. Thus $\pi_1 <\bar{\pi}_1, \bar{\pi}_3> \lambda = \pi_1 j \rho_1 \lambda_1 = j_1 \tau_1 \rho_1 \lambda_1$.

$$\begin{tikzcd}
RoR \arrow[r, tail, "j\#"] \arrow[d, two heads, "\tilde{\tau}"'] & X\times X \arrow[d, "\pi_1"] \\
(RoR^{-1})X \arrow[r, tail, "\beta"'] & X
\end{tikzcd}$$

Since $<\bar{\pi}_1, \bar{\pi}_3> \gamma = j\#\tau\#$ and $\pi_1 j\# = \beta\tilde{\tau}$, the following diagram commutes.

$$\begin{tikzcd}
(R\times X) \cap (X\times R^{-1}) \arrow[r, two heads, "\tilde{\tau}\tau\#"] \arrow[d, "\tau_1\rho_1\lambda_1"'] & (RoR^{-1})X \arrow[d, "\beta"] \arrow[dl, dashed, "\xi"'] \\
RY \arrow[r, tail, "j_1"'] & X
\end{tikzcd}$$

But since $\mathfrak{B}$ has the diagonal property (0.19) and $\tilde{\tau}\tau\#$ is an epimorphism and j_1 is an extremal monomorphism then there exists a morphism ξ such that $j_1\xi = \beta$ and $\tau_1\rho_1\lambda_1 = \xi\tilde{\tau}\tau\#$. Thus $((RoR^{-1})X, \beta) \leq (RY, j_1)$.

Next it will be shown that the following diagram is commutative.

$$\begin{tikzcd}
 & & R\times X \arrow[r, "j\times 1"] & (X\times Y)\times X \arrow[dr, tail, "\theta_1"] & \\
R \arrow[urr, "\langle 1_R,\pi_1 j\rangle"] \arrow[r, dashed, "\xi^*"] \arrow[drr, swap, "\langle \pi_1 j,\tau\rangle"] & (R\times X)\cap(X\times R^{-1}) \arrow[ur, "\lambda_1"] \arrow[dr, swap, "\lambda_2"] \arrow[rrr, tail, "\gamma"] & & & X\times Y\times X \\
 & & X\times R^{-1} \arrow[r, swap, "1\times j^*"] & X\times(Y\times X) \arrow[ur, tail, swap, "\theta_2"] &
\end{tikzcd}$$

$\bar{\pi}_1\theta_1(j\times 1)\langle 1_R,\pi_1 j\rangle = \pi_1\rho_1{}^*(j\times 1)\langle 1_R,\pi_1 j\rangle = \pi_1 j\rho_1\langle 1_R,\pi_1 j\rangle = \pi_1 j.$

$\bar{\pi}_2\theta_1(j\times 1)\langle 1_R,\pi_1 j\rangle = \pi_2\rho_1{}^*(j\times 1)\langle 1_R,\pi_1 j\rangle = \pi_2 j\rho_1\langle 1_R,\pi_1 j\rangle = \pi_2 j.$

$\bar{\pi}_3\theta_1(j\times 1)\langle 1_R,\pi_1 j\rangle = \rho_2{}^*(j\times 1)\langle 1_R,\pi_1 j\rangle = \rho_2\langle 1_R,\pi_1 j\rangle = \pi_1 j.$

$\bar{\pi}_1\theta_2(1\times j^*)\langle \pi_1 j,\tau\rangle = \tilde{\rho}_1(1\times j^*)\langle \pi_1 j,\tau\rangle = \bar{\bar{\pi}}_1\langle \pi_1 j,\tau\rangle = \pi_1 j.$

$\bar{\pi}_2\theta_2(1\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_1\tilde{\rho}_2(1\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_1 j^*\bar{\bar{\pi}}_2\langle \pi_1 j,\tau\rangle = \hat{\pi}_1 j^*\tau =$

$\hat{\pi}_1\langle \pi_2,\pi_1\rangle j = \pi_2 j.$

$\bar{\pi}_3\theta_2(1\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_2\tilde{\rho}_2(1\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_2 j^*\bar{\bar{\pi}}_2\langle \pi_1 j,\tau\rangle = \hat{\pi}_2 j^*\tau =$

$\hat{\pi}_2\langle \pi_2,\pi_1\rangle j = \pi_1 j.$

Consequently there exists a morphism ξ^* such that the above diagram commutes and such that $\gamma\xi^* = \langle \pi_1 j,\pi_2 j,\pi_1 j\rangle$.

Thus $\langle \pi_1,\pi_3\rangle\gamma\xi^* = \langle \pi_1 j,\pi_1 j\rangle$ and hence the following diagram is commutative.

$$\begin{tikzcd}
 & (R\times X)\cap(X\times R^{-1}) \arrow[r, tail, "\gamma"] \arrow[d, two heads, "\tau^{\#}"'] & X\times Y\times X \arrow[d, two heads, "\langle \pi_1,\pi_3\rangle"] \\
 & R\circ R^{-1} \arrow[r, tail, "j^{\#}"] \arrow[d, two heads, "\tilde{\tau}"'] & X\times X \arrow[d, two heads, "\pi_1"] \\
 & (R\circ R^{-1})X \arrow[r, "\theta"] & X \\
R \arrow[uuur, "\xi^*"] \arrow[r, two heads, swap, "\tau_1"] \arrow[urr, "\pi_1 j"] & RY \arrow[ur, tail, swap, "j_1"] &
\end{tikzcd}$$

Since (RY, j_1) is the intersection of all extremal subobjects through which $\pi_1 j$ factors, it follows that $(RY, j_1) \equiv ((R \circ R^{-1})X, \beta)$. Thus $(RY, j_1) \equiv ((R \circ R^{-1})X, \beta)$ which was to be proved.

4.23. Corollary. Let (R, j) be a relation from X to Y. Then (XR, j_2) and $((R^{-1} \circ R)Y, \hat{\beta})$ are isomorphic as extremal subobjects of Y.

Proof. $(XR, j_2) \equiv (R^{-1}X, j_3) \equiv (((R^{-1}) \circ (R^{-1})^{-1})Y, \tilde{\beta}) \equiv ((R^{-1} \circ R)Y, \hat{\beta})$ (4.4, 1.11 and 4.22).

SECTION 5. UNIONS

5.1. Definition. If $\{(R_i,j_i)\colon i\varepsilon I\}$ is a family of relations from X to Y then let $(\overset{*}{\bigcup_{i\varepsilon I}} R_i,j)$ be the intersection of all relations (i.e., extremal subobjects of $X\times Y$) "containing" each (R_i,j_i) (where containment is in the sense of "factoring through" as noted in Remark 0.12). $(\overset{*}{\bigcup_{i\varepsilon I}} R_i,j)$ shall be called the relation theoretic union of the family $\{(R_i,j_i)\colon i\varepsilon I\}$.

5.2. Examples. In the category Set the relation theoretic union is the usual set theoretic union together with the inclusion map.

In the category $\underline{Top}_1$ the relation theoretic union is the usual set theoretic union endowed with the subspace topology determined by the topology of $X\times Y$ together with the inclusion map.

In the category $\underline{Top}_2$ the relation theoretic union is the closure of the set theoretic union together with the inclusion map.

In the categories Grp and Ab the relation theoretic union is the subgroup generated by the set theoretic union of the relations.

5.3. Proposition. Let $\{(R_i,j_i)\colon i\varepsilon I\}$ be a family of relations from X to Y, let $(\bigcup_{i\varepsilon I} R_i,k)$ denote the usual categorical union of subobjects, let $(\sigma,\hat{j})$ be the epi-extremal mono factorization of k and let the codomain of σ (domain of $\hat{j}$) be denoted $\hat{R}$. Then $\hat{R}$ and $\overset{*}{\bigcup_{i\varepsilon I}} R_i$ are isomorphic relations from X to Y.

Proof. Since $(\overset{*}{\bigcup_{i\varepsilon I}} R_i,j)$ is the intersection of all extremal subobjects containing each (R_i,j_i) and each $(R_i,j_i) \leq (\bigcup_{i\varepsilon I} R_i,k)$ and $(\bigcup_{i\varepsilon I} R_i,k) \leq (\hat{R},\hat{j})$ and since $\hat{j}$ is an extremal monomorphism then

$$(\overset{*}{\bigcup_{i\varepsilon I}} R_i, j) \leq (R, j).$$

Since $(\bigcup_{i\varepsilon I} R, k)$ is the intersection of all subobjects which "contain" each (R_i, j_i) then $(\bigcup_{i\varepsilon I} R, k) \leq (\overset{*}{\bigcup_{i\varepsilon I}} R, j)$. Since j is an extremal monomorphism and $(\hat{R}, \hat{j})$ is the intersection of all extremal subobjects which "contain" $(\bigcup_{i\varepsilon I} R, k)$ then $(\hat{R}, \hat{j}) \leq (\overset{*}{\bigcup_{i\varepsilon I}} R, j)$. Thus

$$(\hat{R}, \hat{j}) \equiv (\overset{*}{\bigcup_{i\varepsilon I}} R, j).$$

5.4. <u>Remark</u>. Notice that by the definition of relation theoretic union, if (R_1, j_1), (R_2, j_2), and (S, k) are relations from X to Y and if $(R_1, j_1) \leq (S, k)$ and $(R_2, j_2) \leq (S, k)$, then $(R_1 \overset{*}{\cup} R_2, j) \leq (S, k)$ (cf. 1.19).

5.5. <u>Proposition</u>. Let (R_1, j_1), (R_2, j_2), (S_1, k_1) and (S_2, k_2) be relations from X to Y. If $(R_1, j_1) \leq (R_2, j_2)$ and $(S_1, k_1) \leq (S_2, k_2)$ then

$$(R_1 \overset{*}{\cup} S_1, j) \leq (R_2 \overset{*}{\cup} S_2, k).$$

<u>Proof</u>. $(R_1, j_1) \leq (R_2, j_2) \leq (R_2 \overset{*}{\cup} S_2, k)$ and $(S_1, k_1) \leq (S_2, k_2) \leq (R_2 \overset{*}{\cup} S_2, k)$ whence $(R_1 \overset{*}{\cup} S_1, j) \leq (R_2 \overset{*}{\cup} S_2, k)$ (5.4).

5.6. <u>Remark</u>. The following proposition can be strengthened with the additional hypothesis that the category $\mathfrak{K}$ has finite coproducts (5.34); however, it is included here because it is of interest in its own right.

5.7. <u>Proposition</u>. Let (R, j) be a relation from X to Y and let (S, k) and (T, m) be relations from Y to Z. Then $((RoS) \overset{*}{\cup} (RoT), \beta) \leq (Ro(S \overset{*}{\cup} T), \beta_1)$.

<u>Proof</u>. Consider the following commutative diagrams.

$$\begin{tikzcd}
S \arrow[dr, "\lambda_S"'] \arrow[drr, "k"] & & \\
& S \overset{*}{\cup} T \arrow[r, "\alpha"] & Y \times Z \\
T \arrow[ur, "\lambda_T"] \arrow[urr, "m"'] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & R\times Z \arrow[r, tail, "j\times 1_Z"'] & (X\times Y)\times Z \arrow[dr, tail, "\theta_1"] & \\
(R\times Z)\cap X\times(S\overset{*}{\cup}T) \arrow[ur, "\lambda_3"] \arrow[dr, "\lambda_4"'] \arrow[rrr, tail, "\gamma_1"] & & & X\times Y\times Z \\
 & X\times(S\overset{*}{\cup}T) \arrow[r, tail, "1_X\times\alpha"'] & X\times(Y\times Z) \arrow[ur, tail, "\theta_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & R\times Z \arrow[r, tail, "j\times 1_Z"'] & (X\times Y)\times Z \arrow[dr, tail, "\theta_1"] & \\
R\times Z\cap X\times S \arrow[ur, "\lambda_5"] \arrow[dr, "\lambda_6"'] \arrow[rrr, tail, "\gamma_2"] & & & X\times Y\times Z \\
 & X\times S \arrow[r, tail, "1_X\times k"'] & X\times(Y\times Z) \arrow[ur, tail, "\theta_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & R\times Z \arrow[r, tail, "j\times 1_Z"'] & (X\times Y)\times Z \arrow[dr, tail, "\theta_1"] & \\
R\times Z\cap X\times T \arrow[ur, "\lambda_7"] \arrow[dr, "\lambda_8"'] \arrow[rrr, tail, "\gamma_3"] & & & X\times Y\times Z \\
 & X\times T \arrow[r, tail, "1_X\times m"'] & X\times(Y\times Z) \arrow[ur, tail, "\theta_2"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
R\times Z\cap X\times(S\overset{*}{\cup}T) \arrow[r, tail, "\gamma_1"] \arrow[dr, "\tau_1"'] & X\times Y\times Z \arrow[r, "<\bar{\pi}_1,\bar{\pi}_3>"] & X\times Z \\
 & R\circ(S\overset{*}{\cup}T) \arrow[ur, tail, "\beta_1"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
R\times Z\cap X\times S \arrow[r, tail, "\gamma_2"] \arrow[dr, "\tau_2"'] & X\times Y\times Z \arrow[r, "<\bar{\pi}_1,\bar{\pi}_3>"] & X\times Z \\
 & R\circ S \arrow[ur, tail, "\beta_2"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
(R\times Z)\cap(X\times T) \arrow[r, tail, "\gamma_3"] \arrow[dr, two heads, "\tau_3"'] & X\times Y\times Z \arrow[r, "<\bar{\pi}_1,\bar{\pi}_3>"] & X\times Z \\
& RoT \arrow[ur, tail, "\beta_3"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
RoS \arrow[dr, "\lambda_1"] \arrow[drr, tail, "\beta_2"] & & \\
& (RoS)\mathbin{\overset{*}{\cup}}(RoT) \arrow[r, tail, "\beta"] & X\times Z \\
RoT \arrow[ur, "\lambda_2"] \arrow[urr, tail, "\beta_3"'] & &
\end{tikzcd}$$

Since $(S,k) \leq (S\mathbin{\overset{*}{\cup}}T,\alpha)$ and $(T,m) \leq (S\mathbin{\overset{*}{\cup}}T,\alpha)$ it readily follows that $((R\times Z)\cap(X\times S),\gamma_2) \leq ((R\times Z)\cap(X\times(S\mathbin{\overset{*}{\cup}}T)),\gamma_1)$ and that $((R\times Z)\cap(X\times T),\gamma_3) \leq ((R\times Z)\cap(X\times(S\mathbin{\overset{*}{\cup}}T)),\gamma_1)$. Thus there exist morphisms ξ_1 and ξ_2 such that $\gamma_1\xi_1 = \gamma_2$ and $\gamma_1\xi_2 = \gamma_3$. Hence $<\bar{\pi}_1,\bar{\pi}_3>\gamma_1\xi_1 = <\bar{\pi}_1,\bar{\pi}_3>\gamma_2$ and $<\bar{\pi}_1,\bar{\pi}_3>\gamma_1\xi_2 = <\bar{\pi}_1,\bar{\pi}_3>\gamma_3$.

But (RoS,β_2) is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>\gamma_2$ factors and since $<\bar{\pi}_1,\bar{\pi}_3>\gamma_2 = <\bar{\pi}_1,\bar{\pi}_3>\gamma_1\xi_1 = \beta_1\tau_1\xi_1$, we have $(RoS,\beta_2) \leq (Ro(S\mathbin{\overset{*}{\cup}}T),\beta_1)$. And since (RoT,β_3) is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>\gamma_3$ factors and since $<\bar{\pi}_1,\bar{\pi}_3>\gamma_3 = <\bar{\pi}_1,\bar{\pi}_3>\gamma_1\xi_2 = \beta_1\tau_1\xi_2$ it follows that

$$(RoT,\beta_3) \leq (Ro(S\mathbin{\overset{*}{\cup}}T),\beta_1).$$

Whence $((RoS)\mathbin{\overset{*}{\cup}}(RoT),\beta) \leq (Ro(S\mathbin{\overset{*}{\cup}}T),\beta_1)$ (5.4).

5.8. Proposition. Let (T,m) be a relation from Y to Z and let (R,j) and (S,k) be relations from X to Y. Then

$$((RoT)\mathbin{\overset{*}{\cup}}(SoT),\tilde{\beta}) \leq ((R\mathbin{\overset{*}{\cup}}S)oT,\bar{\beta}).$$

Proof. Analogous to the proof of 5.7.

5.9. Lemma. Let $\{(R_i, j_i) : i \varepsilon I\}$ be a family of relations from X to Y. Then

$$((\bigcup^*_{i\varepsilon I} R_i)^{-1}, j^*) \equiv (\bigcup^*_{i\varepsilon I} (R_i)^{-1}, k).$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
\bigcup^*_{i\varepsilon I} R_i \arrow[rr, "j"] \arrow[drr, "\tau"] & & X\times Y \arrow[rr, "\langle\pi_2,\pi_1\rangle"] & & Y\times X \\
& & (\bigcup^*_{i\varepsilon I} R)^{-1} \arrow[urr, "j^*"] & & \\
R_i \arrow[uu, "\lambda_i"] \arrow[uurr, "j_i"] \arrow[drr, "\tau_i"'] & & & & \\
& & R_i^{-1} \arrow[uuurr, "j_i^*"] \arrow[rr, "\lambda_i^*"'] & & \bigcup^*_{i\varepsilon I} (R_i)^{-1} \arrow[uuu, "k"']
\end{tikzcd}$$

Since (R_i^{-1}, j_i^*) is the intersection of all extremal subobjects through which $\langle\pi_2,\pi_1\rangle j_i$ factors it follows that

$$(R_i^{-1}, j_i^*) \leq ((\bigcup^*_{i\varepsilon I} R_i)^{-1}, j^*).$$

Thus

$$(\bigcup^*_{i\varepsilon I} (R_i^{-1}), k) \leq ((\bigcup^*_{i\varepsilon I} R_i)^{-1}, j^*) \quad (5.4).$$

Now $(R_i, j_i) \leq (\bigcup^*_{i\varepsilon I} (R_i^{-1}), \langle\pi_2,\pi_1\rangle^{-1}k)$ since $\langle\pi_2,\pi_1\rangle^{-1}k\lambda_i^*\tau_i = \langle\pi_2,\pi_1\rangle^{-1}j_i^*\tau_i = j_i$. Thus

$$((\bigcup^*_{i\varepsilon I} R_i), j) \leq ((\bigcup^*_{i\varepsilon I} (R_i^{-1}), \langle\pi_2,\pi_1\rangle^{-1}k)$$

from which it follows that

$$((\bigcup^*_{i\varepsilon I} R_i)^{-1}, j^*) \leq (\bigcup^*_{i\varepsilon I} (R_i^{-1}), \langle\pi_2,\pi_1\rangle\langle\pi_2,\pi_1\rangle^{-1}k)$$

whence

$$((\bigcup^*_{i\varepsilon I} R_i)^{-1}, j^*) \equiv (\bigcup^*_{i\varepsilon I} (R_i^{-1}), k).$$

5.10. Corollary. Let $\{(R_i,j_i)\colon i\varepsilon I\}$ be a family of symmetric relations on X. Then $\bigcup^{*}_{i\varepsilon I} R_i$ is a symmetric relation on X.

Proof. It is clear that for each $i\varepsilon I$, $(R_i,j_i) \equiv (R_i^{-1},j_i^{*})$ (1.13). Thus $(\bigcup^{*}_{i\varepsilon I} R_i,j) \equiv (\bigcup^{*}_{i\varepsilon I}(R_i^{-1}),k) \leq ((\bigcup^{*}_{i\varepsilon I} R_i)^{-1},j^{*})$ (5.9).

5.11. Proposition. If (R,j) is a reflexive relation on X and (S,k) is any relation on X, then $(R\cup^{*}S,m)$ is reflexive on X.

Proof. Since (R,j) is reflexive, $(\Delta_X,i_X) \leq (R,j)$. Thus $(\Delta_X,i_X) \leq (R,j) \leq (R\cup^{*}S,m)$ hence $(R\cup^{*}S,m)$ is reflexive on X.

5.12. Definition. Let (R,j) be a relation on X. Consider the relation $(R\cup^{*}R^{-1},j\#)$.

$$\begin{tikzcd}
R \arrow[dr, "\lambda_R"'] \arrow[drr, tail, "j"] & & \\
& R\cup^{*}R^{-1} \arrow[r, tail, "j\#"] & X\times X \\
R^{-1} \arrow[ur, "\lambda_{R^{-1}}"] \arrow[urr, tail, "j^{*}"'] & &
\end{tikzcd}$$

Let $(\tau\#,\chi)$ be the epi-extremal mono factorization of $\pi_1 j\#$. The domain of χ (codomain of $\tau\#$) shall be denoted by X_R.

$$\begin{tikzcd}
R\cup^{*}R^{-1} \arrow[r, tail, "j\#"] \arrow[d, two heads, "\tau\#"'] & X\times X \arrow[d, "\pi_1"] \\
X_R \arrow[r, tail, "\chi"'] & X
\end{tikzcd}$$

According to the notation of Section 4, X_R is also denoted by

$$(R\cup^{*}R^{-1})X.$$

5.13. <u>Examples</u>. In the category <u>Set</u>, $X_R \cong (R \overset{*}{\cup} R^{-1})X \cong XR \cup RX$. That is, $X_R = \{x \in X\colon \text{there exists } y \in X \text{ such that } (x,y) \in R \text{ or } (y,x) \in R\}$.

In the category <u>Top_1</u>, X_R is the same set as in <u>Set</u> endowed with the subspace topology determined by the topology of X.

5.14. <u>Proposition</u>. Let (R,j) be a relation on X and let (RX,j_1) and $(R^{-1}X,j_3)$ be the images of $\pi_1 j$ and $\pi_1 j^*$ respectively. Then

$$(RX \overset{*}{\cup} R^{-1}X, \alpha) \leq (X_R, \chi) = ((R \overset{*}{\cup} R^{-1})X, \chi).$$

<u>Proof</u>. Consider the following commutative diagram.

$$\begin{tikzcd}
R \arrow[rr, "\tau_1"] \arrow[d, "\lambda_R"'] \arrow[dr, tail, "j"] & & RX \arrow[d, tail, "j_1"] \\
R \overset{*}{\cup} R^{-1} \arrow[r, tail, "j\#"] \arrow[dr, two heads, "\tau\#"'] & X \times X \arrow[r, "\pi_1"] & X \\
& X_R \arrow[ur, tail, "\chi"'] &
\end{tikzcd}$$

Since $\pi_1 j = \pi_1 j\#\lambda_R = \chi\tau\#\lambda_R = j_1\tau_1$ and (RX,j_1) is the intersection of all extremal subobjects through which $\pi_1 j$ factors then $(RX,j_1) \leq (X_R,\chi)$. Similarly, it can be shown that $(R^{-1}X,j_3) \leq (X_R,\chi)$, whence $(RX \overset{*}{\cup} R^{-1}X,\alpha) \leq (X_R,\chi)$ (5.4).

5.15. <u>Proposition</u>. If (R,j) is a relation on X then $(R \overset{*}{\cup} R^{-1}, j\#)$ is symmetric on X and $(X_R,\chi) \equiv (X(R \overset{*}{\cup} R^{-1}), j_2)$.

<u>Proof</u>. $((R \overset{*}{\cup} R^{-1})^{-1}, j\#^*) \equiv (R^{-1} \overset{*}{\cup} (R^{-1})^{-1}, \tilde{j}) \equiv (R^{-1} \overset{*}{\cup} R, j\#)$ (5.9 and 1.11). Thus $(R \overset{*}{\cup} R^{-1}, j\#)$ is symmetric so that

$$(X_R,\chi) = ((R \overset{*}{\cup} R^{-1})X,\chi) \equiv (X(R \overset{*}{\cup} R^{-1}),j_2) \quad (4.6).$$

5.16. <u>Proposition</u>. Let (R,j) be a relation on X and let (Δ_{X_R}, i_{X_R}) be the diagonal of $X_R \times X_R$. Then $(\Delta_{X_R}, (\chi\times\chi)i_{X_R}) \equiv (\Delta_X \cap (X_R \times X_R), \rho)$ where

$(\Delta_X \cap (X_R \times X_R), \rho)$ is the intersection of the diagonal (Δ_X, i_X) of $X \times X$ with $(X_R \times X_R, \chi \times \chi)$.

Proof. Consider the following commutative diagram.

$$
\begin{tikzcd}
\Delta_{X_R} \arrow[rr, tail, "i_{X_R}"] \arrow[dd, dashed, "\xi"'] \arrow[dr, dashed, "\beta"] & & X_R \times X_R \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] \arrow[dd, tail, "\chi \times \chi"] & X_R \arrow[dd, tail, "\chi"] \\
& \Delta_X \cap X_R \times X_R \arrow[ul, dashed, bend right, "\alpha"'] \arrow[ur, "\lambda"] \arrow[dl, "\sigma"'] \arrow[dr, tail, "\rho"] & & \\
\Delta_X \arrow[rr, tail, "i_X"'] & & X \times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X
\end{tikzcd}
$$

Observe that $\pi_1(\chi \times \chi) i_{X_R} = \chi \pi_1^* i_{X_R} = \chi \pi_2^* i_{X_R} = \pi_2(\chi \times \chi) i_{X_R}$. Thus, since i_X is the equalizer of π_1 and π_2, there exists a unique morphism ξ so that $i_X \xi = (\chi \times \chi) i_{X_R}$.

Thus, since $(\Delta_X \cap X_R \times X_R, \rho)$ is the intersection of (Δ_X, i_X) and $(X_R \times X_R, \chi \times \chi)$, there exists a morphism β so that $\rho\beta = i_X \xi = (\chi \times \chi) i_{X_R}$. Consequently $(\Delta_{X_R}, (\chi \times \chi) i_{X_R}) \leq (\Delta_X \cap (X_R \times X_R), \rho)$.

Since $\pi_1 i_X = \pi_2 i_X$ it follows that $\pi_1 \rho = \pi_1 i_X \sigma = \pi_2 i_X \sigma = \pi_2 \rho$, but $(\chi \times \chi)\lambda = \rho$ so that $\pi_1(\chi \times \chi)\lambda = \pi_2(\chi \times \chi)\lambda$. Whence $\chi \pi_1^* \lambda = \chi \pi_2^* \lambda$. Recall that χ is an extremal monomorphism, hence a monomorphism; so that it follows that $\pi_1^* \lambda = \pi_2^* \lambda$. Since (Δ_{X_R}, i_{X_R}) is the equalizer of π_1^* and π_2^* there exists a morphism α such that $i_{X_R} \alpha = \lambda$.

Thus $(\chi \times \chi) i_{X_R} \alpha = (\chi \times \chi)\lambda = \rho$, which means that $(\Delta_X \cap (X_R \times X_R), \rho) \leq (\Delta_{X_R}, (\chi \times \chi) i_{X_R})$. Whence

$$(\Delta_X \cap (X_R \times X_R), \rho) \equiv (\Delta_{X_R}, (\chi \times \chi) i_{X_R}).$$

5.17. Definition. Let (R, j) be a relation on X. Then (R, j) is called

<u>quasi-reflexive</u> if and only if $(\Delta_{X_R}, (\chi\times\chi)i_{X_R}) \leq (R,j)$. That is, (R,j) is quasi-reflexive provided that there exists a morphism λ such that the following diagram commutes.

$$\begin{tikzcd}
\Delta_{X_R} \arrow[r, tail, "i_{X_R}"] \arrow[d, "\lambda"'] & X_R\times X_R \arrow[r, shift left, "\pi_1^*"] \arrow[r, shift right, "\pi_2^*"'] \arrow[d, "\chi\times\chi"'] & X_R \arrow[d, "\chi"] \\
R \arrow[r, tail, "j"'] & X\times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X
\end{tikzcd}$$

5.18. <u>Proposition</u>. If (R,j) is a reflexive relation on X, then (R,j) is quasi-reflexive on X.

<u>Proof</u>. If (R,j) is reflexive then $(R \circledast R^{-1}, j\#)$ is reflexive (5.11); hence, $\pi_1 j\#$ is a retraction (1.24). Thus $\pi_1 j\#$ is an epimorphism; so that if $(\tau\#, \chi)$ is the epi-extremal mono factorization of $\pi_1 j\#$, $\pi_1 j\# = \chi\tau\#$ so that χ is an epimorphism as well as an extremal monomorphism. Hence χ is an isomorphism (0.17). Thus $(X_R, \chi) \equiv (X, 1_X)$ whence $(\Delta_{X_R}, i_{X_R}) \equiv (\Delta_X, i_X)$. Thus, since $(\Delta_X, i_X) \leq (R,j)$, (R,j) is quasi-reflexive.

5.19. <u>Proposition</u>. Let (R,j) be a relation on X. Then $(R,j) \leq (\Delta_X, i_X)$ if and only if $(R,j) \leq (\Delta_{X_R}, (\chi\times\chi)i_{X_R})$.

<u>Proof</u>. Suppose that $(R,j) \leq (\Delta_X, i_X)$. Then there exists a morphism α such that $j = i_X\alpha$. Thus $\pi_1 j = \pi_1 i_X \alpha = \pi_2 i_X \alpha = \pi_2 j$; whence $\pi_1\langle\pi_2,\pi_1\rangle j = \pi_2 j = \pi_1 j = \pi_2\langle\pi_2,\pi_1\rangle j$. Consequently the unique epi-extremal mono factorization of $\langle\pi_2,\pi_1\rangle j$ is $(1_R, j)$, and $(R,j) \equiv (R^{-1}, j^*)$ (3.12).

But, since $(R,j) \equiv (R^{-1}, j^*)$, then $(R \circledast R^{-1}, j\#) \equiv (R,j)$, and $(X_R, \chi) = ((R \circledast R^{-1})X, \chi) \equiv (RX, j_1)$. Thus, since $(RX, j_1) \equiv (X_R, \chi) \equiv (XR, j_2)$ it follows that $(R,j) \leq (X_R\times X_R, \chi\times\chi)$ (3.4).

However, it has been shown that $(\Delta_{X_R}, (\chi\times\chi)i_{X_R})$ and $(\Delta_X \cap (X_R\times X_R), \rho)$ are isomorphic relations on X (5.16). So that since $(R,j) \leq (\Delta_X, i_X)$ and $(R,j) \leq ((X_R\times X_R), \chi\times\chi)$ it follows that

$$(R,j) \leq (\Delta_X \cap (X_R\times X_R), \rho) \equiv (\Delta_{X_R}, (\chi\times\chi)i_{X_R}) \text{ (1.19).}$$

Conversely, if $(R,j) \leq (\Delta_{X_R}, (\chi\times\chi)i_{X_R})$ then since $\pi_1(\chi\times\chi)i_{X_R} = \chi\pi_1{}^*i_{X_R} = \chi\pi_2{}^*i_{X_R} = \pi_2(\chi\times\chi)i_{X_R}$ it follows that $(\Delta_{X_R}, (\chi\times\chi)i_{X_R}) \leq (\Delta_X, i_X)$. Consequently the result $(R,j) \leq (\Delta_X, i_X)$ follows from the transitivity of " $\leq$ " (0.12).

5.20. Theorem. If (R,j) is a relation on X then R, $Ro\Delta_{X_R}$, and $\Delta_{X_R}oR$ are isomorphic relations on X.

Proof. It will be shown that $Ro\Delta_{X_R}$ and R are isomorphic relations on X; the proof for $\Delta_{X_R}oR$ and R is analogous and is omitted.

The following products shall be considered: $(X_R\times X_R; \pi_1{}^*, \pi_2{}^*)$, $(X\times X\times X, \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3)$, $(X\times X, \pi_1, \pi_2)$, $((X\times X)\times X, \tilde{\pi}_1, \tilde{\pi}_2)$, $(X\times(X\times X), \hat{\bar{\pi}}_1, \hat{\bar{\pi}}_2)$, $(R\times X, \tilde{\rho}_1, \tilde{\rho}_2)$ and $(X\times\Delta_{X_R}, \rho_1{}^*, \rho_2{}^*)$.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & R\times X \arrow[r, tail, "j\times 1_X"] & (X\times X)\times X \arrow[d, "\Theta_1"] & \\
R\times X \cap X\times\Delta_{X_R} \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"] \arrow[ddrr, two heads, "\beta"'] \arrow[rr, tail, "\gamma"] & & X\times X\times X \arrow[r, "<\bar{\pi}_1, \bar{\pi}_3>"] & X\times X \\
 & X\times\Delta_{X_R} \arrow[r, tail, "1_X\times(\chi\times\chi)i_{X_R}"] & X\times(X\times X) \arrow[u, "\Theta_2"'] & \\
 & & Ro\Delta_{X_R} \arrow[uur, "\alpha"'] &
\end{tikzcd}$$

Next it will be shown that $<\bar{\pi}_1, \bar{\pi}_2>\gamma = <\bar{\pi}_1, \bar{\pi}_3>\gamma$.

$\pi_1\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_1\gamma = \pi_1\langle\bar\pi_1,\bar\pi_3\rangle\gamma.$

$$\begin{aligned}\pi_2\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_2\gamma &= \bar\pi_2\Theta_2(1_X\times(\chi\times\chi)i_{X_R})\lambda_2' = \pi_1\hat{\bar\pi}_2(1_X\times(\chi\times\chi)i_{X_R})\lambda_2 = \\ &\pi_1(\chi\times\chi)i_{X_R}\rho_2{*}\lambda_2 = \chi\pi_1{*}i_{X_R}\rho_2{*}\lambda_2 = \chi\pi_2{*}i_{X_R}\rho_2{*}\lambda_2 = \\ &\chi\pi_2{*}i_{X_R}\rho_2{*}\lambda_2 = \pi_2(\chi\times\chi)i_{X_R}\rho_2{*}\lambda_2 = \pi_2\hat{\bar\pi}_2(1_X\times(\chi\times\chi)i_{X_R})\lambda_2 = \\ &\bar\pi_3\Theta_2(1_X\times(\chi\times\chi)i_{X_R})\lambda_2 = \bar\pi_3\gamma = \pi_2\langle\bar\pi_1,\bar\pi_3\rangle\gamma.\end{aligned}$$

Thus by the definition of product $\langle\bar\pi_1,\bar\pi_3\rangle\gamma = \langle\bar\pi_1,\bar\pi_2\rangle\gamma$.

Now consider $\langle\bar\pi_1,\bar\pi_2\rangle\gamma$. It will be shown that $\langle\bar\pi_1,\bar\pi_2\rangle\gamma = j\tilde\rho_1\lambda_1$. $\pi_1\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_1\gamma = \bar\pi_1\Theta_1(j\times 1)\lambda_1$. Thus $\pi_1\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \pi_1\tilde\pi_1(j\times 1)\lambda_1 = \pi_1 j\tilde\rho_1\lambda_1$. $\pi_2\langle\bar\pi_1,\bar\pi_2\rangle\gamma = \bar\pi_2\gamma = \bar\pi_2\Theta_1(j\times 1)\lambda_1 = \pi_2\tilde\pi_1(j\times 1)\lambda_1 = \pi_2 j\tilde\rho_1\lambda_1$. Hence, by the definition of product, $\langle\bar\pi_1,\bar\pi_2\rangle\gamma = j\tilde\rho_1\lambda_1$.

Since $(R\circ\Delta_{X_R},\alpha)$ is the intersection of all extremal subobjects through which $\langle\bar\pi_1,\bar\pi_3\rangle\gamma$ factors and $\langle\bar\pi_1,\bar\pi_3\rangle\gamma = \langle\bar\pi_1,\bar\pi_2\rangle\gamma = j\tilde\rho_1\lambda_1$ it follows that $(R\circ\Delta_{X_R},\alpha) \leq (R,j)$.

For the reverse inequality consider $(R\circledast R^{-1},j\#)$. Since $(R\circledast R^{-1},j\#)$ is symmetric, $((R\circledast R^{-1})X,j_1\#)$ and $(X(R\circledast R^{-1}),j_2\#)$ are isomorphic extremal subobjects of X (5.15).

$$\begin{tikzcd}
R\circledast R^{-1} \arrow[r, "j\#"] \arrow[dr, "\xi\#"'] & X\times X \arrow[r, "\pi_2"] & X \\
 & X(R\circledast R^{-1}) \arrow[ur, "\chi\#"] \arrow[dr, dashed, "\xi"] & \\
R \arrow[uu, "\lambda_R"] \arrow[uur, "j"'] & & X_R \arrow[uu, "\chi"']
\end{tikzcd}$$

Thus if $(\xi\#,\chi\#)$ is the epi-extremal mono factorization of $\pi_2 j\#$, there exists an isomorphism ξ such that $\chi\xi = \xi\#$. So in particular, $\pi_2 j\#\lambda_R = \pi_2 j = \chi\#\xi\#\lambda_R = \chi\xi\xi\#\lambda_R$.

Let σ be the isomorphism for which $i_{X_R}\sigma = \langle 1_{X_R}, 1_{X_R}\rangle$ (1.21). It next will be shown that the following equality holds:

$$\langle \pi_1 j, \pi_2 j, \pi_2 j\rangle = \Theta_1\langle j, \pi_2 j\rangle = \Theta_2(1\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle$$

so that the following diagram is commutative.

$$\begin{tikzcd}
 & R\times X \arrow[rr, "j\times 1_X"] & & (X\times X)\times X \arrow[dr, "\Theta_1"] & \\
R \arrow[ur, "\langle 1_R, \pi_2 j\rangle"] \arrow[r, dashed, "\Sigma"] \arrow[dr, "\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle"'] \arrow[rrrr, bend left=50, "\Theta_1\langle j,\pi_2 j\rangle = \langle \pi_1 j, \pi_2 j, \pi_2 j\rangle"] & R\times X \cap X\times\Delta_{X_R} \arrow[u, "\lambda_1"] \arrow[d, "\lambda_2"] \arrow[rrr, "\gamma"] & & & X\times X\times X \\
 & X\times\Delta_{X_R} \arrow[rr, "1_X\times(\chi\times\chi)i_{X_R}"'] & & X\times(X\times X) \arrow[ur, "\Theta_2"'] &
\end{tikzcd}$$

$$\begin{aligned}
\bar{\pi}_1\Theta_2(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle &= \hat{\bar{\pi}}_1(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \\
&\rho_1{}^*\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \pi_1 j.
\end{aligned}$$

$$\begin{aligned}
\bar{\pi}_2\Theta_2(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle &= \pi_1\hat{\bar{\pi}}_2(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \\
&\pi_1(\chi\times\chi)i_{X_R}\rho_2{}^*\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \\
&\chi\rho_1 i_{X_R}\rho_2{}^*\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \chi\rho_1 i_{X_R}\sigma\xi\xi\#\lambda_R = \\
&\chi\rho_1\langle 1_{X_R}, 1_{X_R}\rangle\xi\xi\#\lambda_R = \chi\xi\xi\#\lambda_R = \pi_2 j.
\end{aligned}$$

$$\begin{aligned}
\bar{\pi}_3\Theta_2(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle &= \pi_2\hat{\bar{\pi}}_2(1_X\times(\chi\times\chi)i_{X_R})\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \\
&\pi_2(\chi\times\chi)i_{X_R}\rho_2{}^*\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \\
&\chi\rho_2 i_{X_R}\rho_2{}^*\langle \pi_1 j, \sigma\xi\xi\#\lambda_R\rangle = \chi\rho_2\langle 1_{X_R}, 1_{X_R}\rangle\xi\xi\#\lambda_R = \\
&\chi 1_{X_R}\xi\xi\#\lambda_R = \chi\xi\xi\#\lambda_R = \pi_2 j.
\end{aligned}$$

$$\bar{\pi}_1\Theta_1\langle j, \pi_2 j\rangle = \pi_1\tilde{\pi}_1\langle j, \pi_2 j\rangle = \pi_1 j.$$

$$\bar{\pi}_2\Theta_1\langle j, \pi_2 j\rangle = \pi_2\tilde{\pi}_1\langle j, \pi_2 j\rangle = \pi_2 j.$$

$$\bar{\pi}_3\Theta_1\langle j, \pi_2 j\rangle = \tilde{\pi}_2\langle j, \pi_2 j\rangle = \pi_2 j.$$

Thus by the definition of product:

$$\langle\pi_1 j,\pi_2 j,\pi_2 j\rangle = \Theta_1\langle j,\pi_2 j\rangle = \Theta_2(1_X\times(\chi\times\chi)i_{X_R})\langle\pi_1 j,\sigma\xi\xi\#\lambda_R\rangle.$$

But it is also true that

$$\Theta_1\langle j,\pi_2 j\rangle = \Theta_1(j\times 1)\langle 1_R,\pi_2 j\rangle \text{ (1.31)}.$$

Hence by the definition of intersection there exists a unique morphism Σ such that $\gamma\Sigma = \langle\pi_1 j,\pi_2 j,\pi_2 j\rangle$. Thus

$$\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma = \langle\bar{\pi}_1,\bar{\pi}_3\rangle\langle\pi_1 j,\pi_2 j,\pi_2 j\rangle = \langle\pi_1 j,\pi_2 j\rangle = j.$$

Hence $j = \alpha\beta\Sigma$; and consequently $(R,j) \leq (R\circ\Delta_{X_R},\alpha)$.

5.21. Corollary. Let (R,j) be a relation on X. Then R^{-1}, $\Delta_{X_R}\circ R^{-1}$ and $R^{-1}\circ\Delta_{X_R}$ are isomorphic relations on X.

Proof. Recall that since $(R \circledast R^{-1},j\#)$ is symmetric (5.15), $(X_{R^{-1}},\chi^*)$ and $((R\circledast R^{-1})X,j_1\#)$ are isomorphic as extremal subobjects of X (5.15 and 5.9). Also (X_R,χ) and $((R\circledast R^{-1})X,j_1\#)$ are isomorphic as extremal subobjects of X (5.15) whence,

$(X_{R^{-1}}\times X_{R^{-1}},\chi^*\times\chi^*) \equiv (X_R\times X_R,\chi\times\chi)$ and hence $(\Delta_{X_{R^{-1}}},(\chi^*\times\chi^*)i_{X_{R^{-1}}})$ and $(\Delta_{X_R},(\chi\times\chi)i_{X_R})$. But by the theorem

$$(R^{-1},j^*) \equiv (\Delta_{X_{R^{-1}}}\circ R^{-1},\bar{\alpha}^*) \equiv (R^{-1}\circ\Delta_{X_{R^{-1}}},\alpha^*).$$

Consequently,

$$(R^{-1},j^*) \equiv (\Delta_{X_R}\circ R,\bar{\alpha}\#) \equiv (R^{-1}\circ\Delta_{X_R},\alpha\#).$$

5.22. Definition. Let R be a relation from X to Y. Then R is said to be difunctional if and only if $R\circ(R^{-1}\circ R) \leq R$ and $(R\circ R^{-1})\circ R \leq R$.

The term difunctional relation is due to Riguet [22].

5.23. Proposition. Let R be a relation from X to Y. Then R is difunctional if and only if R^{-1} is difunctional.

Proof. If (R,j) is difunctional then since $(R\circ(R^{-1}\circ R),k_1) \leq (R,j)$ we have $((R^{-1}\circ R)\circ R^{-1},k_1\#) \equiv ((R^{-1}\circ R)^{-1}\circ R^{-1},\tilde{k}_1) \equiv ((R\circ(R^{-1}\circ R))^{-1},k_1^*)\leq(R^{-1},j)$.

This follows from 1.38, 1.11, and 1.12. Also since $((RoR^{-1})oR,k_2) \leq (R,j)$ it follows that

$(R^{-1}o(RoR^{-1}),k_2\#) \equiv (R^{-1}o(RoR^{-1})^{-1},\tilde{k}_2) \equiv (((RoR^{-1})oR)^{-1},k_2{*}) \leq (R^{-1},j{*})$.

Thus $(R^{-1},j{*})$ is difunctional.

If $(R^{-1},j{*})$ is difunctional then since $((R^{-1})^{-1},j\#) \equiv (R,j)$ (1.11) and since, applying the above to $(R^{-1},j{*})$, $((R^{-1})^{-1},j\#)$ is difunctional it follows that (R,j) is difunctional.

5.24. Proposition. If R is a relation on X then R is a quasi-equivalence (3.2) if and only if R is quasi-reflexive and difunctional.

Proof. If (R,j) is a quasi-equivalence then (R,j) is symmetric hence $(R \circledast R^{-1},j\#) \equiv (R,j)$. Thus $(X_R,\chi) \equiv (RX,j_1) \equiv (XR,j_2)$ (5.15 and 4.6) from which it follows that $(\Delta_{X_R},(\chi\times\chi)i_{X_R}) \equiv (\Delta_{XR},(j_2\times j_2)i_{XR})$.

Since (R,j) is a quasi-equivalence then (R,ψ) is an equivalence relation on XR (3.10) so $(\Delta_{XR},i_{XR}) \leq (R,\psi)$. Hence

$(\Delta_{XR},(j_2\times j_2)i_{XR}) \leq (R,j)$ so that (R,j) is quasi-reflexive on X. Since (R,j) is symmetric and transitive then

$(Ro(R^{-1}oR),k_1) \leq (Ro(RoR),\bar{k}_1) \leq (RoR,j') \leq (R,j)$. Similarly, $((RoR^{-1})oR,k_2) \leq ((RoR)oR,\bar{k}_2) \leq (RoR,j') \leq (R,j)$. Hence (R,j) is difunctional.

Conversely if (R,j) is quasi-reflexive and difunctional then

$(\Delta_{X_R},(\chi\times\chi)i_{X_R}) \leq (R,j)$ so that $(\Delta_{X_R},(\chi\times\chi)i_{X_R}) \leq (R^{-1},j{*})$ (1.16 and 1.12). Thus $(RoR,j') \leq (Ro(\Delta_{X_R}oR),\hat{k}) \leq (Ro(R^{-1}oR),k_1) \leq (R,j)$ and

$(R^{-1},j{*}) \leq (\Delta_{X_R}o(R^{-1}o\Delta_{X_R}),\tilde{k}) \leq (Ro(R^{-1}oR),k_1) \leq (R,j)$ (5.20 and 1.30).

Thus (R,j) is both transitive and symmetric hence a quasi-equivalence.

5.25. Proposition. Let R be a relation on X. Then R is an equivalence relation if and only if R is reflexive and difunctional.

Proof. If (R,j) is an equivalence relation on X then (R,j) is reflexive and a quasi-equivalence on X. Thus (R,j) is difunctional (5.18 and 5.24).

Conversely if (R,j) is reflexive and difunctional then (R,j) is quasi-reflexive and difunctional (5.18) hence (R,j) is a quasi-equivalence (5.24). Since (R,j) is also reflexive it must be an equivalence relation.

5.26. Theorem. Let (R,j) be a relation from X to Y and (RX, j_1) and (XR, j_2) be the usual images (3.4). Then R, $Ro\Delta_{XR}$, and $\Delta_{RX}oR$ are isomorphic relations from X to Y (cf. 5.20 and 1.31).

Proof. Consider the following commutative diagrams.

$$\begin{tikzcd}
XR \arrow[r, tail, "j_2"] & Y \\
R \arrow[u, "\tau_2"] \arrow[d, "\tau_1"'] \arrow[r, tail, "j"] & X\times Y \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
RX \arrow[r, tail, "j_1"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta_X \arrow[r, tail, "i_X"] & X\times X \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] & X \\
\Delta_{RX} \arrow[u, dashed, "\alpha"] \arrow[r, tail, "i_{RX}"] & RX\times RX \arrow[u, "j_1\times j_1"'] \arrow[r, shift left, "\rho_1^*"] \arrow[r, shift right, "\rho_2^*"'] & RX \arrow[u, "j_1"']
\end{tikzcd}$$

Observe that $\rho_1(j_1\times j_1)i_{RX} = j_1\rho_1{}^*i_{RX} = j_1\rho_2{}^*i_{RX} = \rho_2(j_1\times j_1)i_{RX}$; thus there exists a morphism α such that $i_X\alpha = (j_1\times j_1)i_{RX}$; i.e.,

$$(\Delta_{RX}, (j_1\times j_1)i_{RX}) \leq (\Delta_X, i_X).$$

Thus: $(\Delta_{RX} \circ R, k_2) \leq (\Delta_X \circ R, j') \equiv (R, j)$ (1.30 and 1.31).

To see that $(R,j) \leq (\Delta_{RX} \circ R, k\ )$ consider the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, "{<\pi_1 j, j>}"] \arrow[dr, "\pi_1"] \arrow[ddr, "j"'] & X\times(X\times Y) \arrow[rr, "\Theta_2"] \arrow[d, "\tilde{\pi}_1"] \arrow[dd, bend left, "\tilde{\pi}_2"] & & X\times X\times Y \arrow[d, "\bar{\pi}_1"'] \arrow[dd, bend left, "{<\bar{\pi}_2, \bar{\pi}_3>}"] \\
& X \arrow[rr, "1_X"] & & X \\
& X\times Y \arrow[rr, "1_{X\times Y}"'] & & X\times Y
\end{tikzcd}$$

Recall that $(\Delta_{RX}, i_{RX}) \equiv (RX, <1_{RX}, 1_{RX}>)$ (1.21) hence there exists an isomorphism σ such that $<1_{RX}, 1_{RX}> = i_{RX}\sigma$.

Consider also the products $(\Delta_{RX}\times Y, \tilde{\rho}_1, \tilde{\rho}_2)$, $(X\times R, \tilde{\pi}_1, \tilde{\pi}_2)$, and $((X\times X)\times Y, \bar{\bar{\pi}}_1, \bar{\bar{\pi}}_2)$.

It will next be shown that the following diagram is commutative.

$$\begin{tikzcd}
& \Delta_{RX}\times Y \arrow[r, "(j_1\times j_1)i_{RX}\times 1_Y"] & (X\times X)\times Y \arrow[dr, "\Theta_1"] & \\
R \arrow[ur, "{<\sigma\tau_1, \pi_2 j>}"] \arrow[rrr, "{<\pi_1 j, \pi_1 j, \pi_2 j> = \Theta_2 <\pi_1 j, j>}"] \arrow[dr, "{<\pi_1 j, 1_R>}"'] & & & X\times X\times Y \\
& X\times R \arrow[r, "i_X\times j"'] & X\times(X\times Y) \arrow[ur, "\Theta_2"'] &
\end{tikzcd}$$

$\bar{\pi}_1 \Theta_1((j_1\times j_1)i_{RX}\times 1_Y)<\sigma\tau_1, \pi_2 j> = \rho_1 \bar{\bar{\pi}}_1((j_1\times j_1)i_{RX}\times 1_Y)<\sigma\tau_1, \pi_2 j> =$

$\rho_1(j_1\times j_1)i_{RX}\tilde{\rho}_1 = j_1\rho_1{}^* i_{RX}\tilde{\rho}_1<\sigma\tau_1, \pi_2 j> = j_1\rho_1{}^* i_{RX}\sigma\tau_1 =$

$j_1\rho_1{}^*<1_{RX}, 1_{RX}>\tau_1 = j_1\tau_1 = \pi_1 j.$

$$\bar{\pi}_2\Theta_1((j_1\times j_1)i_{RX}\times 1_Y)\langle\sigma\tau_1,\pi_2 j\rangle = \rho_2\bar{\bar{\pi}}_1((j_1\times j_1)i_{RX}\times 1_Y)\langle\sigma\tau_1,\pi_2 j\rangle =$$

$$\rho_2(j_1\times j_1)i_{RX}\tilde{\rho}_1\langle\sigma\tau_1,\pi_2 j\rangle = j_1\rho_2{*}i_{RX}\sigma\tau_1 = j_1\rho_2{*}\langle 1_{RX},1_{RX}\rangle\tau_1 =$$

$$j_1\tau_1 = \pi_1 j.$$

$$\bar{\pi}_3\Theta_1((j_1\times j_1)i_{RX}\times 1_Y)\langle\sigma\tau_1,\pi_2 j\rangle = \bar{\bar{\pi}}_2((j_1\times j_1)i_{RX}\times 1_Y)\langle\sigma\tau_1,\pi_2 j\rangle =$$

$$1_Y\tilde{\rho}_2\langle\sigma\tau_1,\pi_2 j\rangle = \pi_2 j.$$

$$\bar{\pi}_1\Theta_2(1_X\times j)\langle\pi_1 j,i_R\rangle = \tilde{\pi}_1(1_X\times j)\langle\pi_1 j,1_R\rangle = \tilde{\tilde{\pi}}_1\langle\pi_1 j,1_R\rangle = \pi_1 j.$$

$$\bar{\pi}_2\Theta_2(1_X\times j)\langle\pi_1 j,1_R\rangle = \pi_1\tilde{\pi}_2(1_X\times j)\langle\pi_1 j,1_R\rangle = \pi_1 j\tilde{\tilde{\pi}}_2\langle\pi_1 j,1_R\rangle = \pi_1 j.$$

$$\bar{\pi}_3\Theta_2(1_X\times j)\langle\pi_1 j,1_R\rangle = \pi_2\tilde{\pi}_2(1_X\times j)\langle\pi_1 j,1_R\rangle = \pi_2 j\tilde{\tilde{\pi}}_2\langle\pi_1 j,1_R\rangle = \pi_2 j.$$

$$\bar{\pi}_1\Theta_2\langle\pi_1 j,j\rangle = \tilde{\pi}_1\langle\pi_1 j,j\rangle = \pi_1 j.$$

$$\bar{\pi}_2\Theta_2\langle\pi_1 j,j\rangle = \pi_1\tilde{\pi}_2\langle\pi_1 j,j\rangle = \pi_1 j.$$

$$\bar{\pi}_3\Theta_2\langle\pi_1 j,j\rangle = \pi_2\tilde{\pi}_2\langle\pi_1 j,j\rangle = \pi_2 j.$$

Consider the following commutative diagram.

$$\begin{tikzcd}
 & \Delta_{RX}\times Y \arrow[r, tail, "(j_1\times j_1)i_{RX}\times 1_Y"] & (X\times X)\times Y \arrow[dr, tail, "\Theta_1"] & & \\
\Delta_{RX}\times Y\cap X\times R \arrow[ur, "\lambda_1"] \arrow[rrr, tail, "\gamma"] \arrow[dr, "\lambda_2"] \arrow[ddrr, bend right, "\tilde{\tau}"'] & & & X\times X\times Y \arrow[r, "\langle\pi_1,\pi_3\rangle"] & X\times Y \\
 & X\times R \arrow[r, tail, "1_X\times j"'] & X\times(X\times Y) \arrow[ur, tail, "\Theta_2"'] & & \\
 & & \Delta_{RX}\circ R \arrow[uurr, tail, "k_2"'] & &
\end{tikzcd}$$

By the definition of intersection, there exists a unique morphism ξ from R to $(\Delta_{RX}\times Y)\cap(X\times R)$ such that $\gamma\xi = \langle\pi_1 j,\pi_1 j,\pi_2 j\rangle$.

Thus $<\bar{\pi}_1,\bar{\pi}_3>\gamma\xi = <\bar{\pi}_1,\bar{\pi}_3><\pi_1 j,\pi_1 j,\pi_2 j> = <\pi_1 j,\pi_2 j> = j$. Hence $k_2\tilde{\tau} = j$; that is, $(R,j) \leq (\Delta_{RX}oR,k_2)$. Hence $(R,j) \equiv (\Delta_{RX}oR,k_2)$.

Similarly it can be shown that $(R,j) \equiv (Ro\Delta_{XR},k_1)$.

5.27. Theorem. Let (R,j) be a relation from X to Y. Then $(\Delta_{RX},(j_1\times j_1)i_{RX}) \leq (RoR^{-1},j\#)$ and $(\Delta_{XR},(j_2\times j_2)i_{XR}) \leq (R^{-1}oR,j')$.

Proof. Consider the following products: $(X\times X,\rho_1,\rho_2)$, $(RX\times RX,\rho_1{*},\rho_2{*})$, $(X\times Y,\pi_1,\pi_2)$, $(R\times X,\bar{\rho}_1,\bar{\rho}_2)$, $(X\times R^{-1},\tilde{\rho}_1,\tilde{\rho}_2)$, $((X\times Y)\times X,\tilde{\pi}_1,\tilde{\pi}_2)$, $(X\times(Y\times X),\pi_1{*},\pi_2{*})$ and $(X\times Y\times X,\bar{\pi}_1,\bar{\pi}_2,\bar{\pi}_3)$.

Also consider the following commutative diagrams.

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[d, two heads, "\tau_1"'] & X\times Y \arrow[d, "\pi_1"] \\
RX \arrow[r, tail, "j_1"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[dr, two heads, "\tau"'] & X\times Y \arrow[r, tail, two heads, "<\pi_2,\pi_1>"] & Y\times X \\
& R^{-1} \arrow[ur, tail, "j^*"'] &
\end{tikzcd}$$

Next, the following diagram will be shown to be commutative.

$$\begin{tikzcd}
& R\times X \arrow[r, tail, "j\times 1_X"] & (X\times Y)\times X \arrow[dr, tail, "\theta_1"] & & \\
R \arrow[ur, "<1_R,\pi_1 j>"] \arrow[r, "\Sigma"] \arrow[dr, "<\pi_1 j,\tau>"'] & (R\times X)\cap(X\times R^{-1}) \arrow[u, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[rr, tail, "\gamma"] \arrow[ddr, bend right, two heads, "\tau\#"'] & & X\times Y\times X \arrow[r, "<\pi_1,\pi_3>"] & X\times X \\
& X\times R^{-1} \arrow[r, tail, "1_X\times j^*"] & X\times(Y\times X) \arrow[ur, tail, "\theta_2"'] & & \\
& & RoR^{-1} \arrow[uurr, tail, "j\#"'] & &
\end{tikzcd}$$

To see this, it need only be observed that

$$\Theta_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \Theta_2(1_X\times j^*)\langle \pi_1 j,\tau\rangle.$$

$$\bar{\pi}_1\Theta_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \pi_1\tilde{\pi}_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \pi_1 j\bar{\rho}_1\langle 1_R,\pi_1 j\rangle = \pi_1 j.$$

$$\bar{\pi}_2\Theta_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \pi_2\tilde{\pi}_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \pi_2 j\bar{\rho}_1\langle 1_R,\pi_1 j\rangle = \pi_2 j.$$

$$\bar{\pi}_3\Theta_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \tilde{\pi}_2(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \bar{\rho}_2\langle 1_R,\pi_1 j\rangle = \pi_1 j.$$

$$\bar{\pi}_1\Theta_2(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \pi_1{}^*(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \tilde{\rho}_1\langle \pi_1 j,\tau\rangle = \pi_1 j.$$

$$\bar{\pi}_2\Theta_2(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_1\pi_2{}^*(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_1 j^*\tilde{\rho}_2\langle \pi_1 j,\tau\rangle = \hat{\pi}_1 j^*\tau =$$
$$\hat{\pi}_1\langle \pi_2,\pi_1\rangle j = \pi_2 j.$$

$$\bar{\pi}_3\Theta_2(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_2\pi_2{}^*(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \hat{\pi}_2\langle \pi_2,\pi_1\rangle j = \pi_1 j.$$

Thus the diagram is commutative and

$$\Theta_1(j\times 1_X)\langle 1_R,\pi_1 j\rangle = \Theta_2(1_X\times j^*)\langle \pi_1 j,\tau\rangle = \langle \pi_1 j,\pi_2 j,\pi_1 j\rangle.$$

Hence by the definition of intersection there exists a morphism Σ such that $\gamma\Sigma = \langle \pi_1 j,\pi_2 j,\pi_1 j\rangle$. Clearly $\langle\bar{\pi}_1,\bar{\pi}_3\rangle\gamma\Sigma = \langle \pi_1 j,\pi_1 j\rangle$ thus $\langle \pi_1 j,\pi_1 j\rangle = j\#\tau\#\Sigma$.

It next will be shown that if σ is that isomorphism for which $\langle 1_{RX}, 1_{RX}\rangle = i_{RX}\sigma$ then $(\sigma\tau_1, (j_1\times j_1)i_{RX})$ is the epi-extremal mono factorization of $\langle \pi_1 j,\pi_1 j\rangle$.

$$\begin{tikzcd}
R \arrow[r, "{\langle \pi_1 j,\pi_1 j\rangle}"] \arrow[d, two heads, "\tau_1"'] & X\times X \\
RX \arrow[r, tail, two heads, "\sigma"'] & \Delta_{RX} \arrow[u, "(j_1\times j_1)i_{RX}"']
\end{tikzcd}$$

$$\rho_1(j_1\times j_1)i_{RX}\sigma\tau_1 = j_1\rho_1{}^*i_{RX}\sigma\tau_1 = j_1\rho_1{}^*\langle 1_{RX},1_{RX}\rangle\tau_1 = j_1\tau_1 = \pi_1 j.$$

$$\rho_2(j_1\times j_1)i_{RX}\sigma\tau_1 = j_1\rho_2{}^*i_{RX}\sigma\tau_1 = j_1\rho_2{}^*\langle 1_{RX},1_{RX}\rangle\tau_1 = j_1\tau_1 = \pi_1 j.$$

Thus the diagram above commutes and $(\sigma\tau_1, (j_1 \times j_1)i_{RX})$ is the epi-extremal mono factorization of $<\pi_1 j, \pi_1 j>$ (0.18). Since $(\Delta_{RX}, (j_1 \times j_1)i_{RX})$ is the intersection of all extremal subobjects through which $<\pi_1 j, \pi_1 j>$ factors and since $<\pi_1 j, \pi_1 j> = j\#\tau\#\Sigma$ it follows that $(\Delta_{RX}, (j_1 \times j_1)i_{RX}) \leq (R \circ R^{-1}, j\#)$ which was to be proved. The proof that $(\Delta_{XR}, (j_2 \times j_2)i_{XR}) \leq (R^{-1} \circ R, j')$ is similar.

5.28. Theorem. If (R,j) is a relation from X to Y then R is difunctional if and only if $(R \circ R^{-1}) \circ R \equiv R \equiv R \circ (R^{-1} \circ R)$.

Proof. If R is difunctional then

$(R,j) \equiv (R \circ \Delta_{XR}, \tilde{k}_1) \leq (R \circ (R^{-1} \circ R), k_1) \leq (R,j)$ (5.26, 5.27 and 5.22).

Similarly,

$(R,j) \equiv (\Delta_{RX} \circ R, \tilde{k}_2) \leq ((R \circ R^{-1}) \circ R, k_2) \leq (R,j)$.

The converse is immediate from the definition (5.22).

5.29. Remark. Let $\mathcal{C}$ be a locally small quasi-complete category having (finite) coproducts. It is noted in passing that if $\mathcal{C}$ has arbitrary products; i.e., is complete, then $\mathcal{C}$ is also (finitely) cocomplete [9]. Recall that the unique epi-extremal mono factorization of a morphism is obtained by taking the intersection of all extremal subobjects of the codomain of the morphism through which the morphism factors (0.21). Also recall that if the intersection of all subobjects of the codomain of the morphism through which the morphism factors is taken, then the unique extremal epi-mono factorization is obtained (0.21). Finally recall that if $\{(A_i, a_i) : i \in I\}$ is a family of subobjects of a $\mathcal{C}$-object X, then the subobject $(\bigcup_{i \in I} A_i, a)$ is obtained by taking the intersection of all subobjects of X which "contain" each (A_i, a_i).

Now consider the coproduct $(\coprod_{i\varepsilon I} A_i, \mu_i)$ of the (finite) family $\{(A_i, a_i)\colon i\varepsilon I\}$. By the definition of coproduct there exists a unique morphism μ such that $\mu\mu_i = a_i$ for each $i\varepsilon I$. Let λ_i be the "inclusion" of (A_i, a_i) into $(\bigcup_{i\varepsilon I} A_i, a)$. Again by the definition of coproduct there exists a unique morphism λ such that $\lambda\mu_i = \lambda_i$ for each $i\varepsilon I$. Thus the following diagram commutes.

$$\begin{tikzcd}
\coprod_{i\varepsilon I} A_i \arrow[rr, "\mu"] \arrow[dr, "\lambda"] & & X \\
& \bigcup_{i\varepsilon I} A_i \arrow[ur, "a"'] & \\
& A_i \arrow[uul, "\mu_i"] \arrow[u, "\lambda_i"'] \arrow[uur, "a_i"'] &
\end{tikzcd}$$

Note that a is a monomorphism. It will be shown that λ is an extremal epimorphism. To see this, it will be shown that $(\bigcup_{i\varepsilon I} A_i, a)$ is the intersection of all subobjects of X through which μ factors. To that end let (Z, g) be any subobject of X through which μ factors; i.e., there is a morphism h such that $\mu = gh$. Then $a_i = \mu\mu_i = gh\mu_i$ hence each (A_i, a_i) factors through (Z, g). Thus by the definition of union there exists a unique morphism ξ such that $g\xi = a$. But this is precisely what is required of the intersection of all subobjects of X through which μ factors.

Now suppose that $\{(A_i, a_i)\colon i\varepsilon I\}$ is a (finite) family of relations from X to Y; i.e., each (A_i, a_i) is an extremal subobject of $X\times Y$. Consider $(\coprod_{i\varepsilon I} A_i, \mu_i)$ and $(\bigcup_{i\varepsilon I} A_i, a)$. Again, let μ be that unique morphism such that $\mu\mu_i = a_i$ for each $i\varepsilon I$. Let λ be that unique morphism such that

$\lambda\mu_i = \lambda_i$; for each $i\varepsilon I$, where λ_i is the inclusion of (A_i, a_i) into $(\bigcup_{i\varepsilon I} A_i, a)$. Let $(\bar{\tau}, \rho)$ be the epi-extremal mono factorization of a. Recalling Proposition 5.3 it follows that the domain of ρ (codomain of $\bar{\tau}$) is $\overset{*}{\bigcup}_{i\varepsilon I} A_i$. Thus the following diagram commutes.

$$\begin{tikzcd}
\coprod_{i\varepsilon I} A_i \arrow[rrr, "\mu"] \arrow[dr, twoheadrightarrow, "\lambda"'] & & & X\times Y \\
& \bigcup_{i\varepsilon I} A_i \arrow[r, tail, twoheadrightarrow, "\bar{\tau}"'] & \overset{*}{\bigcup}_{i\varepsilon I} A_i \arrow[ur, tail, "\rho"'] &
\end{tikzcd}$$

5.30. Theorem. Let $\mathcal{C}$ have (finite) coproducts, let $\{(A_i, a_i): i\varepsilon I\}$ be a (finite) family of subobjects of C, and let f be a $\mathcal{C}$-morphism from C to D. As above, let $(\overset{*}{\bigcup}_{i\varepsilon I} A_i, \rho)$ be the extremal mono part of the factorization of the unique morphism μ from $\coprod_{i\varepsilon I} A_i$ to C with the property that $\mu\mu_i = a_i$ for each $i\varepsilon I$. Let $(\bigcap_{j\varepsilon J} E_j, e)$ be the intersection of all extremal subobjects of D through which each fa_i factors. Let $(\mathrm{Im}(A_i), \rho_i)$ and $(\mathrm{Im}(\overset{*}{\bigcup}_{i\varepsilon I} A_i), \hat{\rho})$ denote the extremal mono parts of the epi-extremal mono factorizations of fa_i and $f\rho$ respectively. Finally let $(\overset{*}{\bigcup}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}})$ be the intersection of all extremal subobjects of D through which each ρ_i factors. Then

$$(\mathrm{Im}(\overset{*}{\bigcup}_{i\varepsilon I} A_i), \hat{\rho}) \equiv (\bigcap_{j\varepsilon J} E_j, e) \equiv (\overset{*}{\bigcup}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}}).$$

Proof. If (E_j, e_j) is an extremal subobject of D through which each fa_i factors, then since $(\mathrm{Im}(A_i), \rho_i)$ is the intersection of all extremal subobjects through which fa_i factors it follows that $(\mathrm{Im}(A_i), \rho_i) \leq (E_j, e_j)$. Thus $(\mathrm{Im}(A_i), \rho_i) \leq (\bigcap_{j\varepsilon J} E_j, e)$ for each $i\varepsilon I$. Hence $(\overset{*}{\bigcup}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}}) \leq (\bigcap_{j\varepsilon J} E_j, e)$.

However, since each fa_i factors through $\hat{\rho}$ and since $\hat{\rho}$ is an extremal monomorphism it follows that $(\bigcap_{j\varepsilon J} E_j, e) \leq (\bigcup^{*}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}})$.

Consider the following commutative diagram.

$$\begin{tikzcd}
 & & & & \bigcup^{*}_{i\varepsilon I} \mathrm{Im}(A_i) \arrow[ddr, tail, "\hat{\hat{\rho}}"] & \\
 & A_i \arrow[dl, "\mu_i"'] \arrow[dr, "a_i"] \arrow[rr, two heads, "\tau_i"] & & \mathrm{Im}(A_i) \arrow[ur, "\lambda_i"] \arrow[drr, tail, "\rho_i"] & & \\
\coprod_{i\varepsilon I} A_i \arrow[rr, "\mu"] \arrow[dr, "\tau"'] \arrow[urrrr, bend left, "\psi"] & & C \arrow[rrr, "f"] & & & D \\
 & \bigcup^{*}_{i\varepsilon I} A_i \arrow[ur, tail, "\rho"'] \arrow[rr, two heads, "\hat{\tau}"'] & & \mathrm{Im}(\bigcup^{*}_{i\varepsilon I} A_i) \arrow[urr, tail, "\hat{\rho}"'] & &
\end{tikzcd}$$

Note that $(\hat{\tau}\tau, \hat{\rho})$ is the epi-extremal mono factorization of $f\mu$ so that $(\mathrm{Im}(\bigcup^{*}_{i\varepsilon I} A_i), \hat{\rho})$ is the intersection of all extremal subobjects through which $f\mu$ factors.

Let ψ be that unique morphism such that $\psi\mu_i = \lambda_i\tau_i$ for each $i\varepsilon I$. Now $\hat{\hat{\rho}}\psi\mu_i = \hat{\hat{\rho}}\lambda_i\tau_i = \rho_i\tau_i = fa_i = f\mu\mu_i$. Thus by the definition of coproduct it follows that $f\mu = \hat{\hat{\rho}}\psi$. Hence $f\mu$ factors through $(\bigcup^{*}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}})$ whence $(\mathrm{Im}(\bigcup^{*}_{i\varepsilon I} A_i), \hat{\rho}) \leq (\bigcup^{*}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}})$.

Now $fa_i = f\mu\mu_i = f\rho\tau\mu_i = \hat{\rho}\hat{\tau}\tau\mu_i$; hence fa_i factors through $\hat{\rho}$, whence $(\bigcap_{j\varepsilon J} E_j, e) \leq (\mathrm{Im}(\bigcup^{*}_{i\varepsilon I} A_i), \hat{\rho})$.

Thus:

$$(\bigcup^{*}_{i\varepsilon I} \mathrm{Im}(A_i), \hat{\hat{\rho}}) \equiv (\bigcap_{j\varepsilon J} E_j, e) \equiv (\mathrm{Im}(\bigcup^{*}_{i\varepsilon I} A_i), \hat{\rho}).$$

5.31. Definition. A category is said to be (finitely) union distributive if the following properties hold:

(i) if X and Y are any $\mathcal{C}$-objects and $\{(A_i,a_i): i\varepsilon I\}$ is a (finite) family of extremal subobjects of Y, then $X\times(\bigcup^*_{i\varepsilon I} A_i)$ and $\bigcup^*_{i\varepsilon I}(X\times A_i)$ are isomorphic relations from X to Y;

(ii) if X is any $\mathcal{C}$-object, $\{(A_i,a_i): i\varepsilon I\}$ is a (finite) family of extremal subobjects of X, and (B,b) is an extremal subobject of X, then $B\cap(\bigcup^*_{i\varepsilon I} A_i)$ and $\bigcup^*_{i\varepsilon I}(B\cap A_i)$ are isomorphic as extremal subobjects of X.

5.32. Remark. It can be shown in any quasi-complete category that $\bigcup^*_{i\varepsilon I}(X\times A_i) \le X\times(\bigcup^*_{i\varepsilon I} A_i)$ and $\bigcup^*_{i\varepsilon I}(B\cap A_i) \le B\cap(\bigcup^*_{i\varepsilon I} A_i)$.

5.33. Examples. It is clear that any union distributive category is finitely union distributive. The categories Set, Top_1, Top_2, and CpT_2 are union distributive.

However, the condition (ii) above is not satisfied in the categories Grp, Ab, SGp^1, and FGp. In fact the condition is not true for a finite family of subobjects. Thus these categories are not finitely union distributive although condition (i) is satisfied.

5.34. Corollary. If $\mathcal{C}$ has (finite) coproducts and is (finitely) union distributive and if $\{(R_i,j_i): i\varepsilon I\}$ is a (finite) family of relations from X to Y and $\{(S_\nu,k_\nu): \nu\varepsilon V\}$ is a (finite) family of relations from Y to Z, then $(\bigcup^*_{i\varepsilon I} R_i)\circ(\bigcup^*_{\nu\varepsilon V} S_\nu)$ and $\bigcup^*_{(i,\nu)\varepsilon I\times V}(R_i\circ S_\nu)$ are isomorphic relations from X to Z. In particular if (R,j) is a relation from X to Y and (S,k) is a relation from Y to Z, then $R\circ(\bigcup^*_{\nu\varepsilon V} S_\nu)$ and $\bigcup^*_{\nu\varepsilon V}(R\circ S_\nu)$ are isomorphic relations from X to Z and $(\bigcup^*_{i\varepsilon I} R_i)\circ S$ and $\bigcup^*_{i\varepsilon I}(R_i\circ S)$ are isomorphic relations from X to Z.

Proof. From the conditions on $\mathcal{C}$ it is easy to see that:

$$
\begin{aligned}
(((\textstyle\bigcup^*_{i\in I} R_i)\times Z)\cap(X\times(\bigcup^*_{\nu\in V} S_\nu)),\tilde{\gamma}) &\equiv (((\textstyle\bigcup^*_{i\in I} R_i)\times Z)\cap(\bigcup^*_{\nu\in V}(X\times S_\nu)),\bar{\gamma}) \\
(\textstyle\bigcup^*_{\nu\in V}(((\bigcup^*_{i\in I} R_i)\times Z)\cap(X\times S_\nu)),\bar{\bar{\gamma}}) &\equiv (\textstyle\bigcup^*_{\nu\in V}((\bigcup^*_{i\in I}(R_i\times Z))\cap(X\times S_\nu),\hat{\gamma}) \\
(\textstyle\bigcup^*_{\nu\in V}(\bigcup^*_{i\in I}((R_i\times Z)\cap(X\times S_\nu))),\hat{\hat{\gamma}}) &\equiv (\textstyle\bigcup^*_{(i,\nu)\in I\times V}((R_i\times Z)\cap(X\times S\nu)),\gamma).
\end{aligned}
$$

Hence, from the theorem (with $(R_i\times Z)\cap(X\times S_\nu)$ assuming the role of A_i) it follows that $((\bigcup^*_{i\in I} R_i)\circ(\bigcup^*_{\nu\in V} S_\nu),\alpha) \equiv (\bigcup^*_{(i,\nu)\in I\times V}(R_i\circ S_\nu),\tilde{\alpha})$.

5.35. Corollary. If $\mathcal{C}$ has finite coproducts and is finitely union distributive and if (R,j) is a quasi-equivalence on X then $(R\bigcup^*\Delta_X,\rho)$ is an equivalence relation on X.

Proof. Clearly $(R\bigcup^*\Delta_X,\rho)$ is both reflexive and symmetric (5.10 and 5.9). Since each of (R,j) and (Δ_X,i_X) is transitive (2.4 and 2.2) it follows that $(R\bigcup^*\Delta_X)\circ(R\bigcup^*\Delta_X),\rho\#) \equiv ((R\circ R)\bigcup^*(R\circ\Delta_X)\bigcup^*(\Delta_X\circ R)\bigcup^*(\Delta_X\circ\Delta_X),\hat{\rho})$

$$\leq (R\textstyle\bigcup^* R\bigcup^* R\bigcup^*\Delta_X,\tilde{\rho}) \equiv (R\bigcup^*\Delta_X,\rho);$$

(5.34 and 1.31). Thus $(R\bigcup^*\Delta_X,\rho)$ is transitive and, consequently, is an equivalence relation.

5.36. Corollary. If $\mathcal{C}$ has (finite) coproducts and is (finitely) union distributive and if (R,j) is a relation from X to Y and $\{(A_i,a_i): i\in I\}$ is a (finite) family of extremal subobjects of X, then $\bigcup^*_{i\in I}(A_iR)$ and $(\bigcup^*_{i\in I}A_i)R$ are isomorphic as extremal subobjects of Y.

Proof. Since $(R\cap((\bigcup^*_{i\in I}A_i)\times Y),\gamma) \equiv (\bigcup^*_{i\in I}(R\cap(A_i\times Y)),\hat{\gamma})$ the result follows from the theorem.

5.37. Corollary. If $\mathcal{C}$ has (finite) coproducts and is (finitely) union distributive and if (R,j) is a relation from X to Y and $\{(B_i,b_i): i\in I\}$ is a (finite) family of extremal subobjects of Y, then $\bigcup^*_{i\in I}(RB_i)$ and $R(\bigcup^*_{i\in I}B_i)$ are isomorphic as extremal subobjects of X.

Proof. Immediate.

5.38. Corollary. If $\mathcal{B}$ has (finite) coproducts and is (finitely) union distributive and if $\{(R_i, j_i) \colon i \in I\}$ is a (finite) family of relations from X to Y and if (A,a) is an extremal subobject of X then $A(\bigcup^{*}_{i\in I} R_i)$ and $\bigcup^{*}_{i\in I}(AR_i)$ are isomorphic as extremal subobjects of Y.

Proof. This result follows immediately from the theorem since

$$((\bigcup^{*}_{i\in I} R_i) \cap (A\times Y), \bar{\gamma}) \equiv (\bigcup^{*}_{i\in I}(R_i \cap (A\times Y)), \tilde{\gamma}).$$

5.39. Corollary. If $\mathcal{B}$ has (finite) coproducts and is (finitely) union distributive and if $\{(R_i, j_i) \colon i \in I\}$ is a (finite) family of relations from X to Y and if (B,b) is an extremal subobject of X then $(\bigcup^{*}_{i\in I} R_i)B$ and $\bigcup^{*}_{i\in I}(R_iB)$ are isomorphic as extremal subobjects of X.

Proof. Immediate.

5.40. Remark. Without the extra conditions on $\mathcal{B}$; i.e., only assuming that $\mathcal{B}$ is locally small and quasi-complete; it is possible to prove that $\bigcup^{*}_{i\in I}(A_iR) \leq (\bigcup^{*}_{i\in I} A_i)R$ and that $\bigcup^{*}_{i\in I}(AR_i) \leq A(\bigcup^{*}_{i\in I} R_i)$.

5.41. Remark. Recall that if g is a $\mathcal{B}$-morphism from X to Y where $\mathcal{B}$ is locally small and quasi-complete then the intersection of all subobjects of Y through which g factors, $(\bigcap_{j\in J} E_j, e)$, yields the extremal epi-mono factorization of g; i.e., there exists an extremal epimorphism h such that $g = eh$. Let $\bigcap_{j\in J} E_j$ be denoted SIm(X) and be called the sub-image of g. (Recall that the image comes from the epi-extremal mono factorization of g.)

$$
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[d, two heads, "h"'] & & Y \\
\mathrm{SIm}(X) \arrow[urr, tail, "e"] \arrow[rr, tail, two heads] & & \mathrm{Im}(X) \arrow[u, tail]
\end{tikzcd}
$$

Now let $\{(A_i,a_i)\colon i\varepsilon I\}$ be a (finite) family of subobjects of a $\mathfrak{C}$-object C and let f be a $\mathfrak{C}$-morphism from C to D. Then there exists a unique morphism μ from the coproduct $(\coprod_{i\varepsilon I} A_i,\mu_i)$ to C such that $\mu\mu_i = a_i$ for each $i\varepsilon I$. Let $\{(E_j,e_j)\colon j\varepsilon J\}$ be the family of all subobjects of D through which each fa_i factors. Let (σ,δ) and (σ_i,δ_i) be the extremal epi-mono factorization of μ and fa_i respectively. Recall that $(\bigcup^*_{i\varepsilon I} \mathrm{SIm}(A_i),\Sigma)$ is the intersection of all subobjects of D through which each δ_i factors. Let (σ^*,δ^*) be the extremal epi-mono factorization of $f\delta$.

5.42. <u>Theorem</u>. If $\mathfrak{C}$ has (finite) coproducts then, using the notation above, $\bigcup_{i\varepsilon I} \mathrm{SIm}(A_i)$, $\bigcap_{j\varepsilon J} E_j$ and $\mathrm{SIm}(\bigcup_{i\varepsilon I} A_i)$ are isomorphic as subobjects of D.

<u>Proof</u>. Let ψ be the unique morphism from $\coprod_{i\varepsilon I} A_i$ to $\bigcup_{i\varepsilon I}\mathrm{SIm}(A_i)$ such that $\psi\mu_i = \gamma_i\sigma_i$ for each $i\varepsilon I$ where γ_i is the "inclusion" of $\mathrm{SIm}(A_i)$ into $\bigcup_{i\varepsilon I}\mathrm{SIm}(A_i)$.

Thus (as is easily seen) the following diagram commutes.

$$
\begin{tikzcd}
 & A_i \arrow[dl, "\mu_i"'] \arrow[rr, two heads, "\sigma_i"] \arrow[dr, "a_i"] & & \mathrm{SIm}(A_i) \arrow[r, "\gamma_i"] \arrow[drr, tail, "\delta_i"'] & \bigcup_{i\varepsilon I}\mathrm{SIm}(A_i) \arrow[dr, "\Sigma"] & \\
\coprod_{i\varepsilon I} A_i \arrow[urrrr, bend left, "\psi"] \arrow[rr, "\mu"] \arrow[dr, two heads, "\sigma"'] & & C \arrow[rrr, "f"] & & & D \\
 & \bigcup_{i\varepsilon I} A_i \arrow[ur, "\delta"'] \arrow[rrr, two heads, "\sigma^*"'] & & & \mathrm{SIm}(\bigcup_{i\varepsilon I} A_i) \arrow[ur, tail, "\delta^*"'] &
\end{tikzcd}
$$

Since fa_i factors through Σ for each $i\varepsilon I$, it follows that

$$(\bigcap_{j\varepsilon J} E_j, e) \leq (\bigcup_{i\varepsilon I} SIm(A_i), \Sigma).$$

Now if (E_j, e_j) is a subobject of D through which each fa_i factors then since $(SIm(A_i), \delta_i)$ is the intersectio of all subobjects of D through which fa_i factors it follows that $(SIm(A_i), \delta_i) \leq (E_j, e_j)$ thus $(\bigcup_{i\varepsilon I} SIm(A_i), \Sigma) \leq (E_j, e_j)$ for each $j\varepsilon J$. Hence $(\bigcup_{i\varepsilon I} SIm(A_i), \Sigma) \leq (\bigcap_{j\varepsilon J} E_j, e)$.

Since each fa_i factors through δ^* then $(\bigcap_{j\varepsilon J} E_j, e) \leq (SIm(\bigcup_{i\varepsilon I} A_i), \delta^*)$. Since $\Sigma\psi = f\mu$ and $\delta^*(\sigma^*\sigma) = f\mu$ is the extremal epi-mono factorization of $f\mu$, it follows that $(SIm(\bigcup_{i\varepsilon I} A_i), \delta^*) \leq (\bigcup_{i\varepsilon I} SIm(A_i), \Sigma)$. Thus

$$(\bigcup_{i\varepsilon I} SIm(A_i), \Sigma) \equiv (\bigcap_{j\varepsilon J} E_j, e) \equiv (SIm(\bigcup_{i\varepsilon I} A_i), \delta^*).$$

5.43. Remark. Theorems 5.30 and 5.42 show that the (sub)image of a union is the union of the (sub)images; hence the epi-extremal mono factorization and the extremal epi- mono factorization properties respect unions in a proper manner.

SECTION 6. RECTANGULAR RELATIONS

6.0. *STANDING HYPOTHESIS. Throughout this section it will be assumed that the category $\mathcal{C}$, in addition to being locally small and quasi-complete, also satisfies the following conditions*:

(i) $\mathcal{C}$ has an initial object Φ such that if X is any $\mathcal{C}$-object then the unique morphism ϕ_X from Φ to X is an extremal monomorphism;

(ii) if X and Y are any $\mathcal{C}$-objects then $(X\times\Phi, 1_X\times\phi_Y) \equiv (\Phi, \phi_{X\times Y})$ and $(\Phi, \phi_{X\times Y}) \equiv (\Phi\times Y, \phi_X\times 1_Y)$;

(iii) projection morphisms (from products other than those isomorphic to the initial object) are epimorphisms.

6.1. *Examples*. The categories $\underline{SGp}$, $\underline{Set}$, $\underline{Top}_1$, $\underline{Top}_2$, and $\underline{CpT}_2$ satisfy the conditions of 6.0. It is also of interest to note that $\underline{Grp}$, $\underline{Ab}$, and $\underline{FGp}$ do not satisfy 6.0 . ii but do satisfy 6.0 . i and 6.0 . iii .

6.2. *Proposition*. If X is any $\mathcal{C}$-object and (A,a) is a subobject of X such that $(A,a) \leq (\Phi, \phi_X)$ then $(A,a) \equiv (\Phi, \phi_X)$.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
\Phi \arrow[d, shift right, "\phi_A"'] \arrow[drr, "\phi_X"] & & \\
A \arrow[u, shift right, "\lambda"'] \arrow[rr, "a"'] & & X
\end{tikzcd}$$

Since there exists a morphism λ for which $\phi_X\lambda = a$ and a is a monomorphism, λ must be a monomorphism. But $\phi_X = \phi_X\lambda\phi_A = \phi_X 1_\Phi$. Thus it follows that $\lambda\phi_A = 1_\Phi$ (6.0.i) so that λ is a retraction. Hence λ is an isomorphism (0. 4) whence $(A,a) \equiv (\Phi,\phi_X)$.

6.3. Proposition. Let (R,j) be a relation from X to Y and let Z be any $\mathcal{E}$-object. Then $R\circ\Phi$ and Φ are isomorphic relations from X to Z; and, $\Phi\circ R$ and Φ are isomorphic relations from Z to Y.

Proof. From 6.0.ii, $(X\times\Phi, 1_X\times\phi_{Y\times Z})$ and $(\Phi,\phi_{X\times(Y\times Z)})$ are isomorphic as extremal subobjects of $X\times(Y\times Z)$ from which it follows that $((R\times Z)\cap(X\times\Phi),\gamma)$ and $(\Phi,\phi_{X\times Y\times Z})$ are isomorphic as extremal subobjects of $X\times Y\times Z$ (6.2). Thus there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd}
(R\times Z)\cap(X\times\Phi) \arrow[rr, "\gamma"] \arrow[drr, "\tau"] \arrow[ddrr, "\sigma"'] & & X\times Y\times Z \arrow[rr, "<\bar\pi_1,\bar\pi_3>"] & & X\times Z \\
& & R\circ\Phi \arrow[urr, "\alpha"] & & \\
& & \Phi \arrow[uurr, "\phi_{X\times Z}"'] & &
\end{tikzcd}$$

Since $\phi_{X\times Z}$ is an extremal monomorphism (6.0.i) and σ is an epimorphism, it follows by the uniqueness of the epi-extremal mono factorization that $(R\circ\Phi,\alpha) \equiv (\Phi,\phi_{X\times Z})$. Similarly it can be shown that $\Phi\circ R$ and Φ are isomorphic relations from Z to Y.

6.4. Corollary. If X is any $\mathcal{E}$-object then $(\Phi,\phi_{X\times X})$ is a quasi-equivalence on X.

<u>Proof</u>. By the proposition $(\Phi,\phi_{X\times X}) \equiv (\Phi o \Phi,\phi_{X\times X})$; hence transitivity is obtained. It is also clear that the following diagram commutes.

$$\begin{tikzcd}
\Phi \arrow[r, "\phi_{X\times X}"] \arrow[dr, "1_\Phi"'] & X\times X \arrow[r, "<\pi_2,\pi_1>"] & X\times X \\
& \Phi \arrow[ur, "\phi_{X\times X}"'] &
\end{tikzcd}$$

Since $\phi_{X\times X}$ is an extremal monomorphism and 1_Φ is an epimorphism it follows from the uniqueness of the epi-extremal mono factorization that $(\Phi,\phi_{X\times X}) \equiv (\Phi^{-1},\phi_{X\times X}*)$; hence symmetry is obtained.

6.5. <u>Definition</u>. Let (R,j) be a relation from X to Y. Then (R,j) is said to be <u>rectangular</u> if and only if there exist extremal subobjects (A,a) and (B,b) of X and Y respectively such that (R,j) and $(A\times B,a\times b)$ are isomorphic relations from X to Y.

6.6. <u>Remark</u>. Since $(\Phi\times\Phi,\phi_X\times\phi_Y)$ and $(\Phi,\phi_{X\times Y})$ are isomorphic as extremal subobjects of $X\times Y$ (6.0.i and 6.0.ii) it follows that $(\Phi,\phi_{X\times Y})$ is a rectangular relation.

6.7. <u>Proposition</u>. Let (R,j) be a rectangular relation from X to Y and let (RY,j_1) and (XR,j_2) be the usual images of $\pi_1 j$ and $\pi_2 j$ respectively. Then R and $RY\times XR$ are isomorphic relations from X to Y.

<u>Proof</u>. Since (R,j) is rectangular there exist extremal subobjects (A,a) and (B,b) of X and Y respectively such that $(R,j) \equiv (A\times B,a\times b)$. Hence there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd}
R \arrow[rr, "j"] \arrow[drr, "\tau_1"] \arrow[dd, "\sigma"'] & & X\times Y \arrow[rr, "\pi_1"] & & X \\
& & RY \arrow[urr, "j_1"] & & \\
A\times B \arrow[rr, "\rho_1"'] & & A \arrow[uurr, "a"'] & &
\end{tikzcd}$$

Since ρ_1 is an epimorphism (6.0.iii), $\rho_1\sigma$ must be an epimorphism, and by the uniqueness of the epi-extremal mono factorization it follows that $(A,a) \equiv (RY,j_1)$. Similarly $(B,b) \equiv (XR,j_2)$. Thus

$$(A\times B, a\times b) \equiv (R,j) \equiv (RY\times XR, j_1\times j_2).$$

6.8. Corollary. Let (R,j) be a relation from X to Y. Then (R,j) is rectangular if and only if (R^{-1},j^*) is rectangular.

Proof. If (R,j) is rectangular then $(R,j) \equiv (RY\times XR, j_1\times j_2)$ (6.7). It is immediate that the following diagram commutes.

$$\begin{tikzcd}
RY\times XR \arrow[rr, "j_1\times j_2"] \arrow[drr, "\langle \pi_2^*, \pi_1^* \rangle"'] & & X\times Y \arrow[rr, "\langle \pi_2, \pi_1 \rangle"] & & Y\times X \\
& & XR\times RY \arrow[urr, "j_2\times j_1"'] & &
\end{tikzcd}$$

Again, by the uniqueness of the epi-extremal mono factorization property it follows that $XR\times RY$ and R^{-1} are isomorphic relations; hence (R^{-1},j^*) is rectangular.

If (R^{-1},j^*) is rectangular then by the above, $((R^{-1})^{-1},j\#)$ is

rectangular; but $((R^{-1})^{-1}, j\#) \equiv (R,j)$ (1.13). Thus (R,j) is rectangular.

6.9. Proposition. Let (R,j) be a rectangular relation from X to Y and let (C,c) be an extremal subobject of X. Then

$$(CR,k) \equiv \begin{cases} (XR, j_2) & \text{if } (C \cap RY, \gamma) \not\equiv (\Phi, \phi_X) \\ (\Phi, \phi_Y) & \text{if } (C \cap RY, \gamma) \equiv (\Phi, \phi_X). \end{cases}$$

Proof. Since $(XR, j_2) \leq (Y, 1_Y)$ it is clear that $(XR \cap Y, \gamma) \equiv (XR, j_2)$. Hence it follows that

$$(R \cap (C \times Y), \gamma_0) \equiv ((RY \times XR) \cap (C \times Y), \gamma_1) \equiv ((RY \cap C) \times (XR \cap Y), \gamma_2) \equiv$$

$$((RY \cap C) \times XR, \beta) \quad (0.8).$$

Thus there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd}
(RY \cap C) \times XR \arrow[r, two heads, "\sigma"] \arrow[rr, bend left, "\beta"] \arrow[ddrr, two heads, "\xi"'] & R \cap (C \times Y) \arrow[r, "\delta"] \arrow[dr, two heads, "\tau"] & X \times Y \arrow[r, "\pi_2"] & Y \\
& & CR \arrow[ur, "k"] & \\
& & Z \arrow[uur, "\Sigma"'] &
\end{tikzcd}$$

Let (ξ, Σ) be the epi-extremal mono factorization of $\pi_2\beta$. Since $\tau\sigma$ is an epimorphism and k is an extremal monomorphism it follows by the uniqueness of the epi-extremal mono factorization that $(Z, \Sigma) \equiv (CR, k)$. But if $((RY \cap C) \times XR, \beta) \not\equiv (\Phi, \phi_{X \times Y})$ then the following diagram commutes.

$$\begin{tikzcd}
(RY \cap C) \times XR \arrow[r, "\beta"] \arrow[d, two heads, "\rho_2"'] & X \times Y \arrow[d, "\pi_2"] \\
XR \arrow[r, "j_2"'] & Y
\end{tikzcd}$$

Since ρ_2 is an epimorphism (6.0.iii) it follows that

$$(XR, j_2) \equiv (Z, \Sigma) \equiv (CR, k).$$

If $((RY \cap C) \times XR, \beta) \equiv (\Phi, \phi_{X\times Y})$ then there exists an isomorphism λ such that the following diagram commutes.

$$\begin{tikzcd}
\Phi \arrow[r, "\lambda"] \arrow[drr, "1_\Phi"'] & (RY \cap C)\times XR \arrow[r, "\beta"] & X\times Y \arrow[r, "\pi_2"] & Y \\
& & \Phi \arrow[ur, "\phi_Y"'] &
\end{tikzcd}$$

Thus by the uniqueness of the epi-extremal mono factorization it follows that $(\Phi, \phi_Y) \equiv (Z, \Sigma) \equiv (CR, k)$.

6.10. Proposition. Let (R,j) be a rectangular relation from X to Y and let (A,a) be an extremal subobject of Y. Then

$$(RA, k) \equiv \begin{cases} (RY, j_1) & \text{if } (XR \cap A, \gamma) \not\equiv (\Phi, \phi_Y) \\ (\Phi, \phi_X) & \text{if } (XR \cap A, \gamma) \equiv (\Phi, \phi_Y). \end{cases}$$

Proof. The proof is analogous to that of Proposition 6.9.

6.11. Proposition. Let (R,j) be a rectangular relation from X to Y and let (S,k) be a relation from Y to Z. Then $RoS \leq RY\times(XR)S$.

Proof. It is easy to see that the following objects are isomorphic as extremal subobjects of $X\times Y\times Z$: $((RY\times XR)\times Z)\cap(X\times S)$, $((RY\times XR)\times Z)\cap(RY\times S)$, and $RY\times((XR\times Z)\cap S)$. Thus there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd}
RY\times((XR\times Z)\cap S) \arrow[r, "\sigma"] & ((RY\times XR)\times Z)\cap(X\times S) \arrow[r, "<\bar{\pi}_1, \bar{\pi}_3>\gamma"] \arrow[dr, "\tau"'] & X\times Z \\
& & (RY\times XR)oS \arrow[u, "\tilde{\alpha}"']
\end{tikzcd}$$

Recall that $(R,j) \equiv (RY\times XR, j_1\times j_2)$ (6.7); hence $(RY\times XR)oS$ and RoS are isomorphic relations. Consider the following commutative diagrams.

$$\begin{tikzcd}
(XR\times Z)\cap S \arrow[r, two heads, "\delta_1"] \arrow[dr, two heads, "\tau_1"'] & Y\times Z \arrow[r, "\tilde{\rho}_2"] & Z \\
& (XR)S \arrow[ur, tail, "\alpha_1"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
RY\times((XR\times Z)\cap S) \arrow[r, tail, "1_{RY}\times\delta_1"] \arrow[dr, two heads, "\tilde{\tau}"] \arrow[ddr, "1_{RY}\times\tau_1"'] & RY\times(Y\times Z) \arrow[r, "\Theta_2(j_1\times 1_{Y\times Z})"] & X\times Y\times Z \arrow[r, "<\bar{\pi}_1,\bar{\pi}_3>"] & X\times Z \\
& P \arrow[urr, tail, "\rho"] & & \\
& RY\times(XR)S \arrow[uurr, tail, "j_1\times\alpha_1"'] & &
\end{tikzcd}$$

Let $(\tilde{\tau},\rho)$ be the epi-extremal mono factorization of

$$<\bar{\pi}_1,\bar{\pi}_3>\Theta_2(j_1\times 1_{Y\times Z})(1_{RY}\times\delta_1) = <\bar{\pi}_1,\bar{\pi}_3>\gamma\sigma.$$

Thus since (P,ρ) is the intersection of all extremal subobjects through which $<\bar{\pi}_1,\bar{\pi}_3>\gamma\sigma$ factors and since σ is an isomorphism it follows that $((RoS),\alpha) \equiv ((RY\times XR)oS,\tilde{\alpha}) \equiv (P,\rho) \leq (RY\times(XR)S, j_1\times\alpha_1)$.

6.12. Proposition. Let (R,j) be a relation from X to Y and let (S,k) be a rectangular relation from Y to Z. Then $RoS \leq R(SZ)\times YS$.

Proof. Since $(S,k) \equiv (SZ\times YS, k_1\times k_2)$ (6.7) the result follows from arguments analogous to those in the proof of Proposition 6.11.

6.13. Lemma. Let $(R_1,j_1) \equiv (A\times B_1, a\times b_1)$ be a rectangular relation from X to Y and let $(R_2,j_2) \equiv (B_2\times C, b_2\times c)$ be a rectangular relation from Y to Z.

Then

$$(R_1 o R_2, j') \equiv ((A\times B_1)o(B_2\times C), j\#) \equiv \begin{cases} (A\times C, a\times c) & \text{if } (B_1 \cap B_2, b) \not\equiv (\Phi, \phi_Y) \\ (\Phi, \phi_{X\times Z}) & \text{if } (B_1 \cap B_2, b) \equiv (\Phi, \phi_Y). \end{cases}$$

And in either case $(R_1 o R_2, j')$ is rectangular.

<u>Proof</u>. It is straightforward (but tedious) to show that:

$$\begin{aligned} ((A\times B_1)\times Z)\cap(X\times(B_2\times C)) &\equiv (A\times(B_1\times Z))\cap(X\times(B_2\times C)) \equiv (A\times(B_1\times Z)\cap(A\times(B_2\times Z)) \\ &\equiv A\times((B_1\times Z)\cap(B_2\times C)) \equiv A\times((B_1\cap B_2)\times C) \\ &\equiv A\times(B_1\cap B_2)\times C. \end{aligned}$$

If $(B_1 \cap B_2, b) \not\equiv (\Phi, \phi_Y)$ then there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd} ((A\times B_1)\times Z)\cap(X\times(B_2\times C)) \arrow[r, "\sigma"] \arrow[ddr, "\tau\#"'] & A\times(B_1\cap B_2)\times C \arrow[r, "\delta"] \arrow[dr, "<\bar\rho_1,\bar\rho_3>"'] & X\times Y\times Z \arrow[r, "<\bar\pi 1,\bar\pi 3>"] & X\times Z \\ & & A\times C \arrow[ur, "a\times c"] & \\ & (A\times B_1)o(B_2\times C) \arrow[uurr, bend right, "j\#"'] & & \end{tikzcd}$$

It is easy to prove that $<\bar\rho_1,\bar\rho_3>$ is an epimorphism since $A\times(B_1\cap B_2)\times C$ and $(A\times C)\times(B_1\cap B_2)$ are isomorphic in a canonical way and 6.0.iii holds. Thus since σ is an isomorphism it follows by the uniqueness of the epi-extremal mono factorization that

$$((A\times B_1)o(B_2\times C), j\#) \equiv (A\times C, a\times c).$$

Hence $(R_1 o R_2, j') \equiv (A\times C, a\times c)$; so it is rectangular.

If $(B_1 \cap B_2, b) \equiv (\Phi, \phi_Y)$ then $(A\times(B_1\cap B_2)\times C, \delta)$ and $(\Phi, \phi_{X\times Y\times Z})$ are isomorphic as extremal subobjects hence by the uniqueness of the epi-extremal mono factorization it follows that $((A\times B_1)o(B_2\times C), j\#) \equiv (\Phi, \phi_{X\times Z})$. Thus $(R_1 o R_2, j') \equiv (\Phi, \phi_{X\times Z})$ and is rectangular (6.6).

6.14. Remark. As has been noted in Section 1 (1.37) the composition of relations is not necessarily associative. The examples 1.34 - 1.36 are in the category $\underline{Top}_2$ which satisfies the conditions of 6.0. The next theorem shows that the composition of rectangular relations is associative. Hence. in particular, in $\underline{Top}_2$ the composition of rectangular relations is associative.

6.15. Theorem. Let (R,j) be a rectangular relation from X to Y, let (S,k) be a rectangular relation from Y to Z and let (T,m) be a rectangular relation from Z to W. Then $Ro(SoT)$ and $(RoS)oT$ are isomorphic relations from X to W.

Proof. Since each of (R,j), (S,k) and (T,m) is rectangular there exist extremal subobjects of X, Y, Z and W such that $(R,j) \equiv (A_1 \times A_2, a_1 \times a_2)$, $(S,k) \equiv (B_1 \times B_2, b_1 \times b_2)$ and $(T,m) \equiv (C_1 \times C_2, c_1 \times c_2)$. Then:

$$(SoT, k\#) \equiv \begin{cases} (B_1 \times C_2, b_1 \times c_2) & \text{if } (B_2 \cap C_1, b) \not\equiv (\Phi, \phi_Z) \\ (\Phi, \phi_{Y \times W}) & \text{if } (B_2 \cap C_1, b) \equiv (\Phi, \phi_Z), \end{cases}$$

and

$$(RoS, j\#) \equiv \begin{cases} (A_1 \times B_2, a_1 \times b_2) & \text{if } (A_2 \cap B_1, a) \not\equiv (\Phi, \phi_Y) \\ (\Phi, \phi_{X \times Z}) & \text{if } (A_2 \cap B_1, a) \equiv (\Phi, \phi_Y). \end{cases}$$

Thus there are two cases:

1) if $(SoT, k\#) \equiv (B_1 \times C_2, b_1 \times c_2)$ then as above it follows from 6.13 that

$$(Ro(SoT), \alpha) \equiv \begin{cases} (A_1 \times C_2, a_1 \times c_2) & \text{if } (A_2 \cap B_1, a) \not\equiv (\Phi, \phi_Y) \\ (\Phi, \phi_{X \times W}) & \text{if } (A_2 \cap B_1, a) \equiv (\Phi, \phi_Y). \end{cases}$$

If $(A_2 \cap B_1, a) \not\equiv (\Phi, \phi_Y)$ then $(RoS, j\#) \equiv (A_1 \times B_2, a_1 \times b_2)$ hence $((RoS)oT, \beta) \equiv (A_1 \times C_2, a_1 \times c_2)$ since $(B_2 \cap C_1, b) \not\equiv (\Phi, \phi_Z)$.

If $(A_2 \cap B_1, a) \equiv (\Phi, \phi_Y)$ then $(RoS, j\#) \equiv (\Phi, \phi_{X \times Z})$ hence $((RoS)oT, \beta) \equiv (\Phi, \phi_{X \times W})$ (6.3).

2) If $(SoT,k\#) \equiv (\Phi,\phi_{X\times W})$ then $(Ro(SoT),\alpha) \equiv (\Phi,\phi_{X\times W})$ (6.3). If $(RoS,j\#) \equiv (A_1\times B_2,a_1\times b_2)$ then $((RoS)oT,\beta) \equiv (\Phi,\phi_{X\times W})$ since $(B_2\cap C_1,b) \equiv (\Phi,\phi_Z)$ (6.13).

If $((RoS),j\#) \equiv (\Phi,\phi_{X\times Z})$ then $((RoS)oT,\beta) \equiv (\Phi,\phi_{X\times W})$ (6.3).

Thus in any case $(Ro(SoT),\alpha) \equiv ((RoS)oT,\beta)$.

6.16. Proposition. Let $\{(R_i,j_i): i\varepsilon I\}$ be a family of rectangular relations from X to Y. Then $(\bigcap_{i\varepsilon I} R_i,j)$ is a rectangular relation from X to Y.

Proof. Each (R_i,j_i) is isomorphic to $(A_i\times B_i,a_i\times b_i)$ where by the definition of rectangular relation (A_i,a_i) and (B_i,b_i) are extremal subobjects of X and Y respectively.

But $(\bigcap_{i\varepsilon I}(A_i\times B_i),\gamma) \equiv ((\bigcap_{i\varepsilon I} A_i)\times(\bigcap_{i\varepsilon I} B_i),a\times b)$ (0. 8). Thus $(\bigcap_{i\varepsilon I} R_i,j) \equiv (\bigcap_{i\varepsilon I}(A_i\times B_i),\gamma) \equiv ((\bigcap_{i\varepsilon I} A_i)\times(\bigcap_{i\varepsilon I} B_i),a\times b)$ which says that $(\bigcap_{i\varepsilon I} R_i,j)$ is a rectangular relation.

6.17. Proposition. Let (R,j) be a symmetric relation on X and let (A_1,a_1) and (A_2,a_2) be extremal subobjects of X such that $(A_1\times A_2,a_1\times a_2) \leq (R,j)$. Then $(A_2\times A_1,a_2\times a_1) \leq (R,j)$.

Proof. Since it is evident that the following diagram is commutative, it follows that $((A_1\times A_2)^{-1},(a_1\times a_2)^*)$ and $(A_2\times A_1,a_2\times a_1)$ are isomorphic relations on X.

$$\begin{tikzcd}
A_1\times A_2 \arrow[r, "a_1\times a_2"] \arrow[dr, "\langle\rho_2,\rho_1\rangle"'] & X\times X \arrow[r, "\langle\pi_2,\pi_1\rangle"] & X\times X \\
& A_2\times A_1 \arrow[ur, "a_2\times a_1"'] &
\end{tikzcd}$$

Thus, since (R,j) is symmetric and $(A_1\times A_2,a_1\times a_2) \leq (R,j)$, it fol-

lows that $(A_2 \times A_1, a_2 \times a_1) \equiv ((A_1 \times A_2)^{-1}, (a_1 \times a_2)^*) \leq (R^{-1}, j^*) \equiv (R,j)$ (1.13 and 1.12).

6.18. Definition. Let (R,j) be a relation from X to Y and let (A_1,a_1) and (A_2,a_2) be extremal subobjects of X and Y respectively. Then $(A_1 \times A_2, a_1 \times a_2)$ is said to be a maximal rectangle in R if and only if $(A_1 \times A_2, a_1 \times a_2) \leq (R,j)$ and if (B_1,b_1) and (B_2,b_2) are extremal subobjects of X and Y respectively such that $(A_1 \times A_2, a_1 \times a_2) \leq (B_1 \times B_2, b_1 \times b_2) \leq (R,j)$, then $(B_1 \times B_2, b_1 \times b_2) \equiv (A_1 \times A_2, a_1 \times a_2)$.

6.19. Proposition. Let $\mathfrak{C}$ be finitely union distributive, let (R,j) be a difunctional relation from X to Y (5.22), and let (R_1,k_1) and (R_2,k_2) be maximal rectangles in R such that $(R_1,k_1) \not\equiv (R_2,k_2)$. Then $(R_1Y \cap R_2Y, \mu) \equiv (\Phi, \phi_X)$ and $(XR_1 \cap XR_2, \lambda) \equiv (\Phi, \phi_Y)$. Hence, in particular, $(R_1 \cap R_2, \gamma) \equiv (\Phi, \phi_{X \times Y})$.

Proof. If $(R_1,k_1) \equiv (\Phi, \phi_{X \times Y})$ then $(R_1,k_1) \leq (R_2,k_2)$ since the following diagram commutes.

$$\begin{tikzcd} R_2 \arrow[r, "k_2"] & X \times Y \\ \Phi \arrow[u, "\phi_{R_2}"] \arrow[ur, "\phi_{X \times Y}"'] & \end{tikzcd}$$

Thus, since (R_1,k_1) is a maximal rectangle, it follows that $(R_1,k_1) \equiv (R_2,k_2)$ contradicting the hypothesis. Hence $(R_1,k_1) \not\equiv (\Phi, \phi_{X \times Y})$. Similarly $(R_2,k_2) \not\equiv (\Phi, \phi_{X \times Y})$.

Now, $(R_1,k_1) \equiv (R_1Y \times XR_1, \mu_1 \times \mu_2)$ and $(R_2,k_2) \equiv (R_2Y \times XR_2, \lambda_1 \times \lambda_2)$ (6.7). By an argument similar to that used in the proof of Proposition 6.17 it follows that $(R_2^{-1}, k_2^*) \equiv (XR_2 \times R_2Y, \lambda_2 \times \lambda_1)$, and

$(R_1{}^{-1},k_1{}^*) \equiv (XR_1 \times R_1Y, \mu_2 \times \mu_1)$.

Suppose that $(XR_1 \cap XR_2, \lambda) \not\equiv (\Phi, \phi_Y)$. Then $(R_1 o R_2{}^{-1}, \tilde{j}) \equiv (R_1Y \times R_2Y, \mu_1 \times \lambda_1)$ (6.13) and hence $((R_1 o R_2{}^{-1}) o R_2, \alpha)$ and $(R_1Y \times XR_2, \mu_1 \times \lambda_2)$ are isomorphic relations (6.13). Similarly $(R_2 o R_1{}^{-1}, \tilde{k}) \equiv (R_2Y \times R_1Y, \lambda_1 \times \mu_1)$ (6.13) and hence $((R_2 o R_1{}^{-1}) o R_1, \beta) \equiv (R_2Y \times XR_1, \lambda_1 \times \mu_2)$ (6.13).

Since $(R_1,k_1) \leq (R,j)$ and $(R_2,k_2) \leq (R,j)$, $(R_1{}^{-1},k_1{}^*) \leq (R^{-1},j^*)$ and $(R_2{}^{-1},k_2{}^*) \leq (R^{-1},j^*)$ (1.12) and hence $((R_1 o R_2{}^{-1}) o R_2, \alpha) \leq ((R o R^{-1}) o R, j')$ and $((R_2 o R_1{}^{-1}) o R_1, \beta) \leq ((R o R^{-1}) o R, j')$ (1.30). Hence

$(R_1 \cup^* R_2 \cup^* ((R_1 o R_2{}^{-1}) o R_2) \cup^* ((R_2 o R_1{}^{-1}) o R_1), \Sigma_1) \leq (R \cup^* ((R o R^{-1}) o R), \Sigma_2)$

(5.5). Since (R,j) is difunctional (5.22), $(R \cup^* ((R o R^{-1}) o R), \Sigma_2) \leq (R,j)$ (5.5). Thus since $\mathfrak{E}$ is finitely union distributive, it follows that:

$((R_1Y \cup^* R_2Y) \times (XR_1 \cup^* XR_2), \xi) \equiv$

$((R_1Y \times XR_1) \cup^* (R_2Y \times XR_2) \cup^* (R_1Y \times XR_2) \cup^* (R_2Y \times XR_1), \tilde{\Sigma}) \equiv$

$(R_1 \cup^* R_2 \cup^* ((R_1 o R_2{}^{-1}) o R_2) \cup^* ((R_2 o R_1{}^{-1}) o R_1), \Sigma) \leq$

(R,j).

Let $K = (R_1Y \cup^* R_2Y) \times (XR_1 \cup^* XR_2)$. Since $(R_1,k_1) \leq (K,\xi)$ and $(R_2,k_2) \leq (K,\xi)$ and (K,ξ) is rectangular, by the definition of maximal rectangle it follows that $(R_1,k_1) \equiv (K,\xi) \equiv (R_2,k_2)$, contradicting the hypothesis. Thus $(XR_1 \cap XR_2, \lambda) \equiv (\Phi, \phi_Y)$.

Now suppose that $(R_1Y \cap R_2Y, \mu) \not\equiv (\Phi, \phi_X)$. Then $(R_1{}^{-1} o R_2, \tilde{\tilde{j}}) \equiv (XR_1 \times XR_2, \mu_2 \times \lambda_2)$ and $((R_1 o (R_1{}^{-1} o R_2), \tilde{\alpha}) \equiv (R_1Y \times XR_2, \mu_1 \times \lambda_2)$ (6.13). Similarly $(R_2{}^{-1} o R_1, \tilde{\tilde{k}}) \equiv (XR_2 \times XR_1, \lambda_1 \times \mu_2)$ and $(R_2 o (R_2{}^{-1} o R_1), \tilde{\beta}) \equiv (R_2Y \times XR_1, \lambda_1 \times \mu_2)$ (6.13). Thus $(K,\xi) \equiv (R_1 \cup^* R_2 \cup^* (R_1 o (R_1{}^{-1} o R_2)) \cup^* (R_2 o (R_2{}^{-1} o R_1)), \Sigma_2)$. Hence $(K,\xi) \leq (R \cup^* (R o (R^{-1} o R)), \tilde{\Sigma}) \leq (R,j)$. Again since (K,ξ) is rectangular

and since $(R_1,k_1) \leq (K,\xi)$ and $(R_2,k_2) \leq (K,\xi)$ it follows that $(R_1,k_1) \equiv (K,\xi) \equiv (R_2,k_2)$ contradicting the hypothesis. Consequently, $(R_1Y \cap R_2Y,\mu) \equiv (\Phi,\phi_X)$.

The above implies that $(R_1 \cap R_2,\gamma) \equiv (\Phi,\phi_{X\times Y})$ since

$$((R_1Y\times XR_1) \cap (R_2Y\times XR_2),\tilde{\gamma}) \equiv ((R_1Y \cap R_2Y)\times(XR_1 \cap XR_2),\mu\times\lambda) \equiv$$

$$(\Phi\times\Phi,\phi_X\times\phi_Y) \equiv (\Phi,\phi_{X\times Y}) \text{ (0.8 and 6.6).}$$

6.20. Proposition. If $\mathcal{C}$ has (finite) coproducts and is (finitely) union distributive and if $\{(A_i\times B_i,a_i\times b_i)\colon i\varepsilon I\}$ is a (finite) family of rectangular relations from X to Y such that $(A_i \cap A_j,a) \equiv (\Phi,\phi_X)$ and $(B_i \cap B_j,b) \equiv (\Phi,\phi_Y)$ for $i \neq j$, $i,j\varepsilon I$ then $(R,j) = (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times B_i),j)$ is a difunctional relation from X to Y.

Proof. First consider (R^{-1},j^*). Since

$((A_i\times B_i)^{-1},(a_i\times b_i)^*) \equiv (B_i\times A_i,b_i\times a_i)$ it follows that

$(R^{-1},j^*) = ((\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times B_i))^{-1},j^*) \equiv (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(B_i\times A_i),\tilde{j})$ (5.8). Thus

$(RoR^{-1},\alpha) \equiv (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times B_i)o\underset{i\varepsilon I}{\overset{*}{\bigcup}}(B_i\times A_i),\tilde{\alpha})$. But

$(\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times B_i)o\underset{i\varepsilon I}{\overset{*}{\bigcup}}(B_i\times A_i),\tilde{\alpha}) \equiv (\underset{(i,j)\varepsilon I\times I}{\overset{*}{\bigcup}}((A_i\times B_i)o(B_j\times A_j)),\tilde{\tilde{\alpha}})$ (5.34). From this and the fact that $(B_i \cap B_j,b) \equiv (\Phi,\phi_Y)$ for $i \neq j$, $i,j\varepsilon I$ it follows that

$(RoR^{-1},\alpha) \equiv (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times A_i),\alpha)$ (6.13).

Similarly $((RoR^{-1})oR,k_1) \equiv (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times A_i)o\underset{j\varepsilon I}{\overset{*}{\bigcup}}(A_j\times B_j),\tilde{\gamma})$

$$(\underset{(i,j)\varepsilon I\times I}{\overset{*}{\bigcup}}((A_i\times A_i)o(A_j\times B_j)),\gamma) \text{ (5.34).}$$

But since $(A_i \cap A_j,a) \equiv (\Phi,\phi_X)$ for $i \neq j$, $i,j\varepsilon I$ it follows that

$(\underset{(i,j)\varepsilon I\times I}{\overset{*}{\bigcup}}((A_i\times A_i)o(A_j\times B_j)),\gamma) \equiv (\underset{i\varepsilon I}{\overset{*}{\bigcup}}(A_i\times B_i),j) = (R,j)$. Thus it has been shown that $((RoR^{-1})oR,k_1) \equiv (R,j)$. Similarly it can be shown that $(Ro(R^{-1}oR),k_2) \equiv (R,j)$. Hence (R,j) is difunctional.

6.21. Definitions. Let X be any $\mathcal{C}$-object and let X be a relation on X. Then (R,j) is a square in $X\times X$ if and only if there exists an extremal

subobject (A,a) of X such that R and $A\times A$ are isomorphic relations on X.

If (S,k) is a relation on X and (R,j) is a square in $X\times X$ such that $(R,j) \leq (S,k)$ then (R,j) is said to be a maximal square in (S,k) if and only if for any square (T,m) in $X\times X$ for which $(R,j) \leq (T,m) \leq (S,k)$ holds, it follows that $(T,m) \equiv (R,j)$.

6.22. Proposition. Let $\mathcal{C}$ be finitely union distributive and let (S,k) be a quasi-equivalence on X. Then (R,j) is a maximal rectangle in (S,k) if and only if (R,j) is a maximal square in (S,k).

Proof. Assume (R,j) is a maximal square in (S,k). Suppose (R,j) is not a maximal rectangle in (S,k) then there exist extremal subobjects (B,b) and (C,c) of X such that $(R,j) \leq (B\times C, b\times c) \leq (S,k)$ and $(R,j) \not\equiv (B\times C, b\times c)$.

Since (S,k) is symmetric and $(B\times C, b\times c) \leq (S,k)$, $(C\times B, c\times b) \leq (S^{-1}, k^*) \equiv (S,k)$ (1.12 and 1.13). Since (S,k) is transitive it follows that

$((B\times C)o(C\times B), \alpha) \equiv (B\times B, b\times b) \leq (SoS^{-1}, \bar{\alpha}) \equiv (SoS, k\#) \leq (S,k)$ and

$((C\times B)o(B\times C), \tilde{\alpha}) \equiv (C\times C, c\times c) \leq (S^{-1}oS, \bar{\bar{\alpha}}) \equiv (SoS, k\#) \leq (S,k)$ (6.13 and 1.30).

Since $\mathcal{C}$ is finitely union distributive it follows that

$$(B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C) \equiv ((B \overset{*}{\cup} C)\times B) \overset{*}{\cup} ((B \overset{*}{\cup} C)\times C)$$
$$(B\times B) \overset{*}{\cup} (C\times B) \overset{*}{\cup} (B\times C) \overset{*}{\cup} (C\times C).$$

Since each of $B\times B$, $C\times B$, $B\times C$, and $C\times C$ is contained in S it follows that $(R,j) \leq (B\times C, b\times c) \leq ((B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C), \beta) \leq (S,k)$. Since $(R,j) \not\equiv (B\times C, b\times c)$, $(R,j) \not\equiv ((B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C), \beta)$ contradicting the maximality of (R,j).

Conversely if (R,j) is a maximal rectangle in (S,k) then $(R,j) \equiv (B\times C, b\times c)$ for some pair of extremal subobjects (B,b) and (C,c)

of X. Repeating the above it follows that $(R,j) \equiv (B\times C, b\times c)$ and $(B\times C, b\times c) \leq ((B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C), \beta) \leq (S,k)$. Since $((B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C), \beta)$ is a rectangular relation then by the maximality of (R,j) it follows that $(R,j) \equiv ((B \overset{*}{\cup} C)\times(B \overset{*}{\cup} C), \beta)$ and hence is a square. Thus (R,j) is a maximal square in (S,k).

6.23. Example. Consider the following symmetric relation in $\underline{\text{Top}}_1$. Let $X = [0,1]$ with the usual topology, let
$S = [0,1] \times [\frac{1}{4}, 3/4] \cup [\frac{1}{4}, 3/4] \times [0,1]$ and let k be the inclusion map taking S into $X\times X$.

It is clear that $[0,1] \times [\frac{1}{4}, 3/4]$ together with its inclusion map is a maximal rectangle in (S,k) that is not a maximal square. It is also clear that $[\frac{1}{4}, 3/4] \times [\frac{1}{4}, 3/4]$ together with its inclusion map is a maximal square in (S,k) that is not a maximal rectangle.

Note this shows that even in a symmetric relation it may be the case that both maximal squares and maximal rectangles exist and are distinct.

Also note that the above example is valid in $\underline{\text{Top}}_2$ and in $\underline{\text{CpT}}_2$. By neglecting the topology and considering the underlying set, the example is valid in Set.

6.24. Proposition. If $\mathcal{B}$ is finitely union distributive, (R,j) is a quasi-equivalence on X, and (R_1,k_1) and (R_2,k_2) are maximal squares in (R,j) such that $(R_1,k_1) \not\equiv (R_2,k_2)$ then $(R_1 \cap R_2, k) \equiv (\Phi, \phi_{X\times X})$.

Proof. Both of (R_1,k_1) and (R_2,k_2) are maximal rectangles (6.23). (R,j) is difunctional (5.24) hence the result follows immediately from 6.19.

6.25. Proposition. Let $\mathcal{B}$ have (finite) coproducts and be (finitely) union distributive and let $\{(A_i,a_i) : i\varepsilon I\}$ be a (finite) family of

extremal subobjects of X with the property that $(A_i \cap A_j, \bar{a}) \equiv (\Phi, \phi_X)$ for $i \neq j$, $i,j \varepsilon I$. Then $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is a quasi-equivalence on X.

Proof. Since $(A_i \times A_i, a_i \times a_i) \equiv ((A_i \times A_i)^{-1}, (a_i \times a_i)^*)$ (3.3) it follows that $((\bigcup^*_{i\varepsilon I}(A_i \times A_i))^{-1}, a^*) \equiv (\bigcup^*_{i\varepsilon I}(A_i \times A_i)^{-1}, \hat{a}) \equiv (\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ (5.8). Hence $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is symmetric.

Next observe that:

$(\bigcup^*_{i\varepsilon I}(A_i \times A_i) \circ \bigcup^*_{i\varepsilon I}(A_i \times A_i), a\#) \equiv (\bigcup^*_{(i,j)\varepsilon I \times I}((A_i \times A_i) \circ (A_j \times A_j)), \beta)$ (5.34). Hence,

since $(A_i \cap A_j, \bar{a}) \equiv (\Phi, \phi_X)$ for $i \neq j$, it follows that

$(\bigcup^*_{(i,j)\varepsilon I \times I}((A_i \times A_i) \circ (A_j \times A_j)), \beta) \equiv (\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ (6.13). Thus transitivity is obtained and $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is a quasi-equivalence on X.

6.26. Definition. Let X be a $\mathcal{B}$-object and let $\{(A_i, a_i) : i\varepsilon I\}$ be a (finite) family of extremal subobjects of X for which $(A_i \cap A_j, \bar{a}) \equiv (\Phi, \phi_X)$ if $i \neq j$, $i,j \varepsilon I$. Such a family is said to be a (finite) partition of X if $(\bigcup^*_{i\varepsilon I} A_i, a) \equiv (X, 1_X)$.

6.27. Theorem. Let $\mathcal{B}$ have (finite) coproducts and be (finitely) union distributive and let $\{(A_i, a_i) : i\varepsilon I\}$ be a (finite) partition of X. Then $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is an equivalence relation on X.

Proof. In view of 6.25 it is immediate that $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is a quasi-equivalence on X. To see that $(\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ is reflexive it suffices to show that $\pi_1 a$ is an epimorphism (3. 9).

Since $(A_i \times A_i, a_i \times a_i)$ is rectangular it follows that $((A_i \times A_i)X, j_1) \equiv (A_i, a_i) \equiv (X(A_i \times A_i), j_2)$ (6.7). Also since $(A_i \times A_i, a_i \times a_i) \leq (\bigcup^*_{i\varepsilon I}(A_i \times A_i), a)$ it follows that for each $i\varepsilon I$, $((A_i \times A_i)X, j_1) \leq ((\bigcup^*_{i\varepsilon I}(A_i \times A_i))X, a_1)$ (4.10); i.e., for each $i\varepsilon I$,

$(A_i,a_i) \leq ((\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i))X,a_1)$. Thus $(\mathop{\circledast}\limits_{i\varepsilon I} A_i,\tilde{a}) \leq ((\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i))X,a_1)$ (5.1).
But $((\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i))X,a_1) \leq (X,1_X)$ and $(\mathop{\circledast}\limits_{i\varepsilon I} A_i,\tilde{a}) \equiv (X,1_X)$. Thus
$(X,1_X) \equiv (\mathop{\circledast}\limits_{i\varepsilon I} A_i,a) \equiv ((\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i))X,a_1)$. That is, there exists an isomorphism σ such that the following diagram commutes.

$$\begin{tikzcd}
\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i) \arrow[rr, "a"] \arrow[drr, "\tau_1"] \arrow[ddrr, "\pi_1 a"'] & & X\times X \arrow[rr, "\pi_1"] & & X \\
& & (\mathop{\circledast}\limits_{i\varepsilon I}(A_i\times A_i))X \arrow[urr, "a"] & & \\
& & X \arrow[u] \arrow[uurr, "1_X"'] & &
\end{tikzcd}$$

Thus $\pi_1 a = \sigma^{-1}\tau_1$. But τ_1 is an epimorphism and σ is an isomorphism; hence $\pi_1 a$ is an epimorphism, as was to be proved.

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